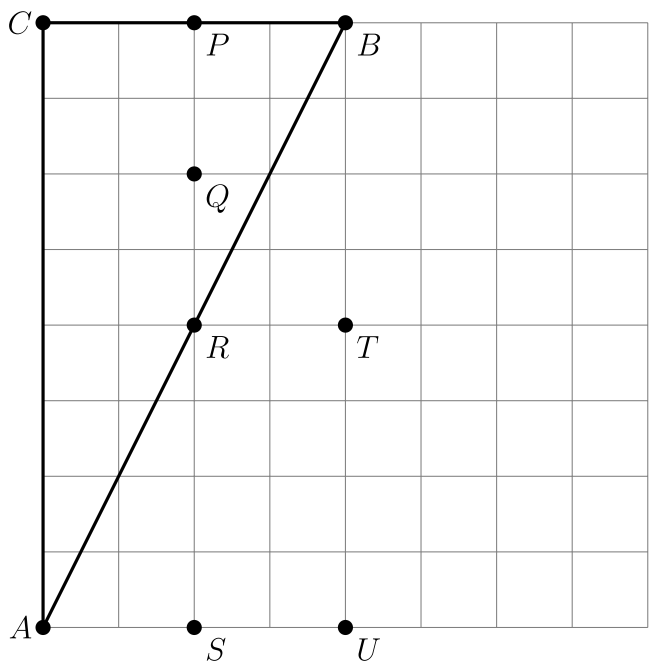
CS 130, Final

Solutions

Read the entire exam before beginning. **Manage your time carefully.** This exam has 60 points; you need 48 to get full credit. Additional points are extra credit. 60 points → 3.0 min/point. 48 points → 3.8 min/point.

Problem 1 (3 points)

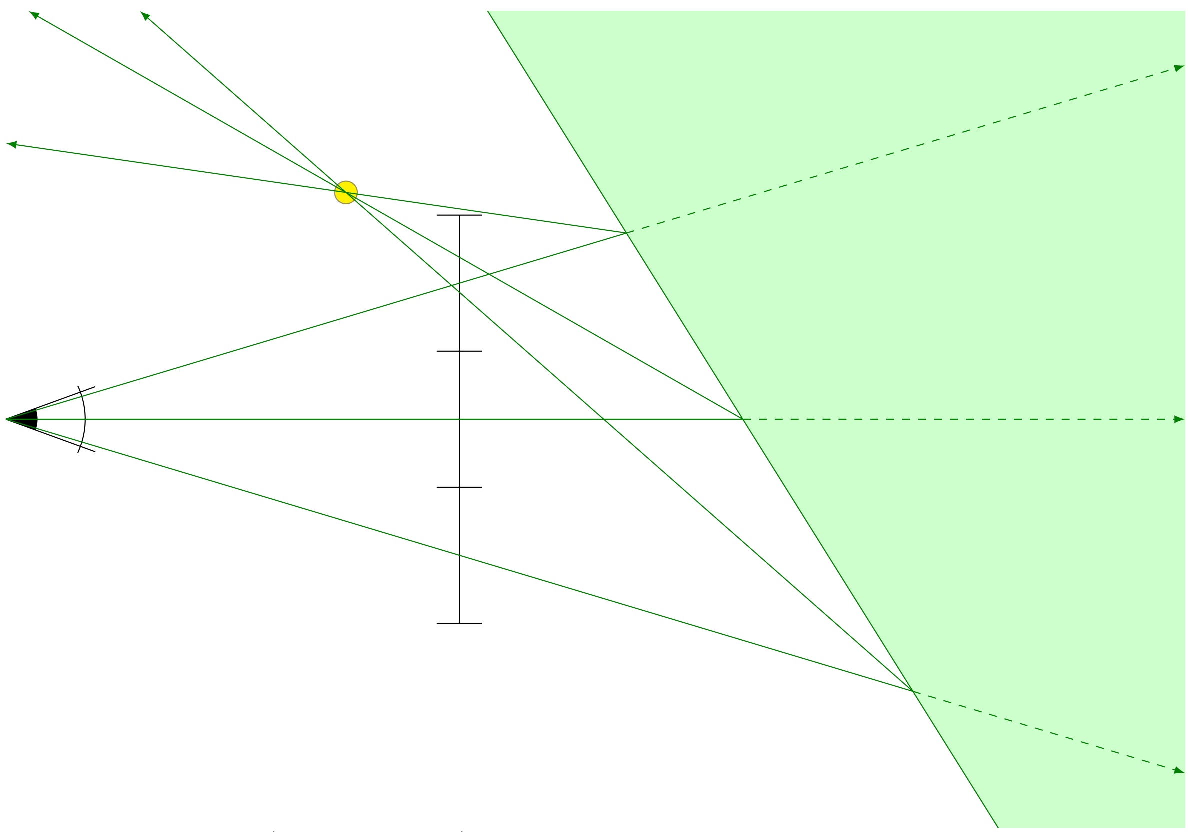
For each set of barycentric weights below, label the corresponding point in the figure with those weights. You do not need to show your work, but your points must be plotted accurately to count.



Point	α	β	γ
A	1	0	0
B	0	1	0
C	0	0	1
P	0	$\frac{1}{2}$	$\frac{1}{2}$
Q	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
R	$\frac{1}{2}$	$\frac{1}{2}$	0
S	1	$\frac{1}{2}$	$-\frac{1}{2}$
T	$\frac{1}{2}$	1	$-\frac{1}{2}$
U	1	1	-1

In the raytracing problems below, green objects are wood, red objects are reflective, and blue objects are transparent. The scenes are in 2D with a 1D image. yellow circles are point lights; the ray tracer supports shadows. Draw all of the rays that would be cast while raytracing each scene. Use a maximum recursion depth of 5. (Don't worry about precisely what counts as depth 5; I just care that recursion is being performed correctly when necessary and that important rays are not missing. There are no more than 20 rays in the "exact" solution.)

Problem 2 (4 points)

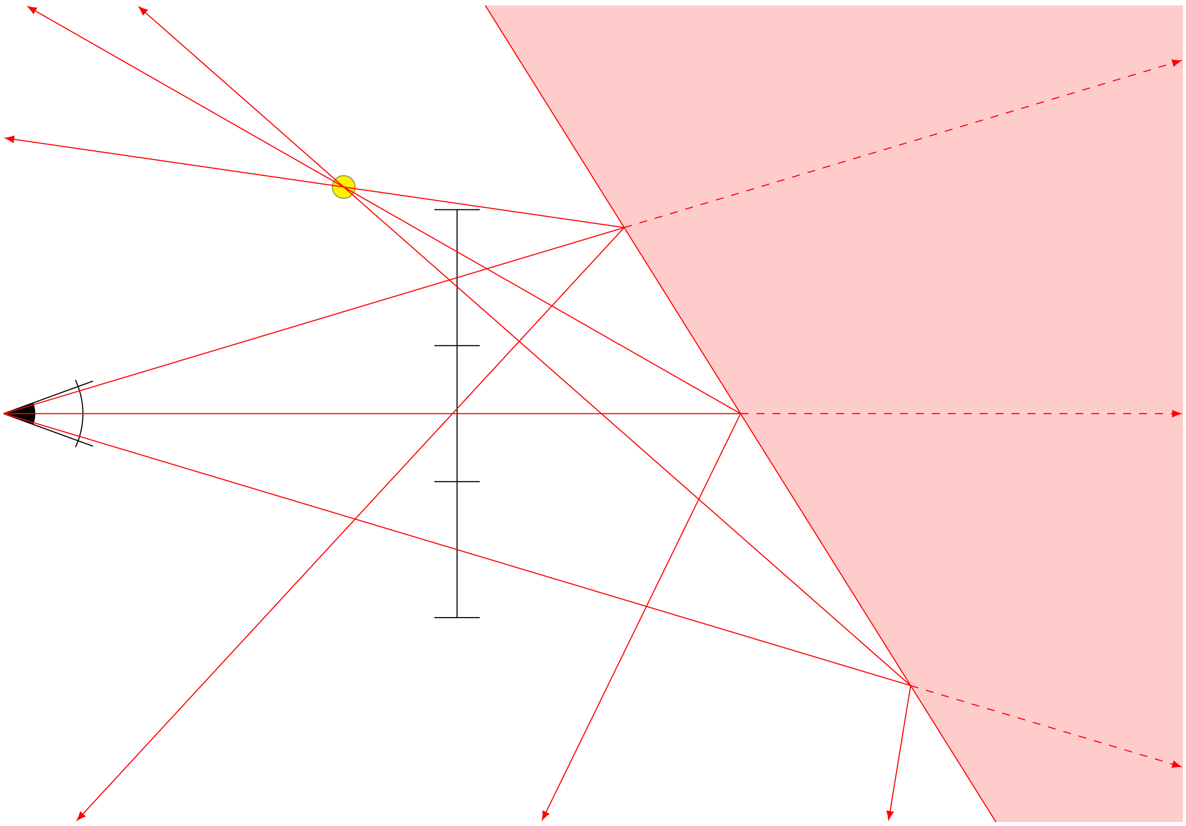


Problem 3 (2 points)

The clipper tends to remove the significant majority of triangles. Since the vertex shader can be quite expensive, running the vertex shader after clipping could result in a significant performance boost. Nevertheless, this is never done. Why do we run the vertex shader before performing clipping?

The vertex shader transforms vertices into normalized device coordinates. Since the triangles need to be in normalized device coordinates in order to do clipping, it is not possible to clip before running the vertex shader.

Problem 4 (4 points)



Problem 5 (2 points)

Given two unit vectors $\vec{u}$ and $\vec{w}$, construct a unit vector $\vec{q}$ orthogonal to both of them.

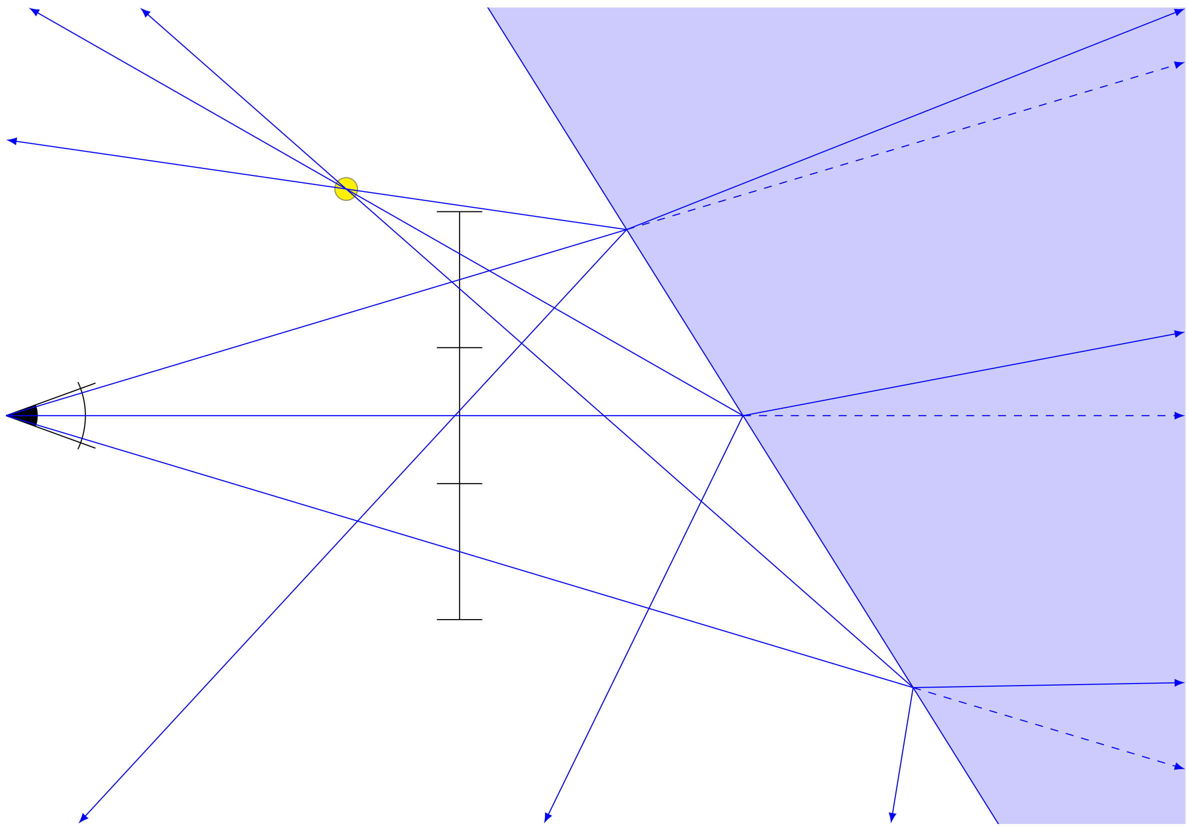
$$\vec{q} = \frac{\vec{u} \times \vec{w}}{\|\vec{u} \times \vec{w}\|}$$

Problem 6 (2 points)

Construct a normalized vector that is orthogonal to the 2D vector $\langle x, y \rangle$.

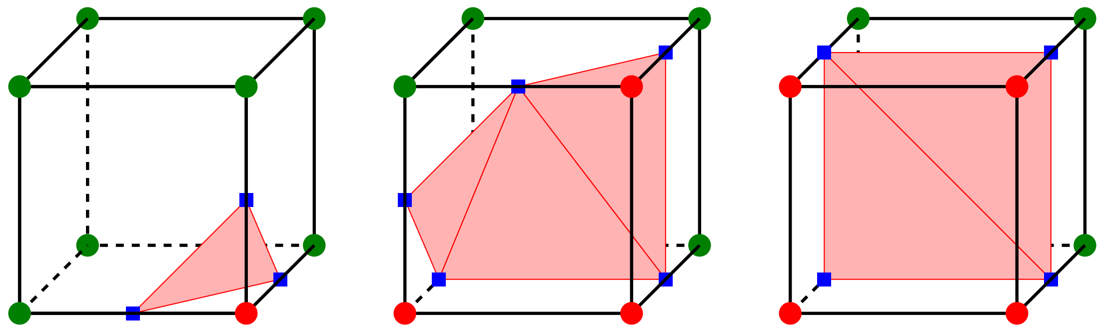
$\begin{pmatrix} -\frac{y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \end{pmatrix}$ or its negation.

Problem 7 (4 points)



Problem 8 (2 points)

Construct a consistent marching cubes triangulation of the cubes shown below. The cubes should actually be touching, but they have been separated apart for clarity. Draw squares at the vertices of your triangles.



Problem 9 (3 points)

Let $f(x,y,z) = m^4 - 2(k^2 + x^2 + y^2 - z^2)m^2 + (k^2 - x^2 - y^2 - z^2)^2$. The surface $f(x,y,z) = 0$ defines a torus (donut shape) with big radius m and small radius k (with $m > k$) centered at the origin with the z axis as its axis of rotation. Determine the **distance** from a camera at location $(m, 0, 0)$ to the nearest point on the torus along the direction $\mathbf{u} = \langle 0, 0, 1\rangle$. *This problem should come out nicely; if it does not, check your work.*

$$
\begin{aligned}
&x = m \qquad y = 0 \qquad z = t \\
&0 = m^4 - 2(k^2 + x^2 + y^2 - z^2)m^2 + (k^2 - x^2 - y^2 - z^2)^2 \\
&0 = m^4 - 2(k^2 + m^2 - t^2)m^2 + (k^2 - m^2 - t^2)^2 \\
&0 = m^4 - 2m^2k^2 - 2m^4 + 2m^2t^2 + k^4 + m^4 + t^4 - 2k^2m^2 - 2k^2t^2 + 2m^2t^2 \\
&0 = t^4 + 2(2m^2 - k^2)t^2 - k^2(4m^2 - k^2) \\
&t^2 = \frac{-2(2m^2 - k^2) \pm \sqrt{4(2m^2 - k^2)^2 + 4k^2(4m^2 - k^2)}}{2} \\
&t^2 = -2m^2 + k^2 \pm \sqrt{4m^4 - 4m^2k^2 + k^4 + 4k^2m^2 - k^4} \\
&t^2 = -2m^2 + k^2 \pm \sqrt{4m^4} \\
&t^2 = -2m^2 + k^2 \pm 2m^2 \\
&t^2 = -4m^2 + k^2 \qquad \text{or} \qquad t^2 = k^2
\end{aligned}
$$

Noting that $-4m^2 + k^2 < 0$, the only solutions are $t = \pm k$. Since we must have $t \geq 0$, the distance is k.

Problem 10 (2 points)

What is the normal direction at an arbitrary point (x, y, z) lying on the surface of the torus? *Don't worry about whether the normal points inwards or outwards. You do not need to normalize the normal direction for this problem.*

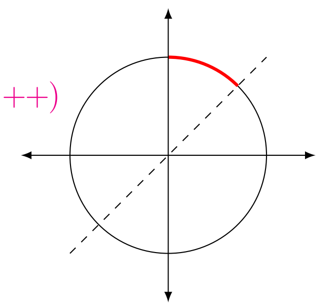
$$
\begin{aligned}
\mathbf{n} = \nabla f &= \begin{pmatrix} \frac{\partial}{\partial x}(m^4 - 2(k^2 + x^2 + y^2 - z^2)m^2 + (k^2 - x^2 - y^2 - z^2)^2) \\ \frac{\partial}{\partial y}(m^4 - 2(k^2 + x^2 + y^2 - z^2)m^2 + (k^2 - x^2 - y^2 - z^2)^2) \\ \frac{\partial}{\partial z}(m^4 - 2(k^2 + x^2 + y^2 - z^2)m^2 + (k^2 - x^2 - y^2 - z^2)^2) \end{pmatrix} \\
&= \begin{pmatrix} -4xm^2 - 4x(k^2 - x^2 - y^2 - z^2) \\ -4ym^2 - 4y(k^2 - x^2 - y^2 - z^2) \\ +4zm^2 - 4z(k^2 - x^2 - y^2 - z^2) \end{pmatrix}
\end{aligned}
$$

The problem statement does not require us to normalize.

Problem 11 (6 points)

In this problem, we will adapt the midpoint algorithm to rasterize the outline of a circle. Assume first that the circle has integer radius r and is centered at the origin. Lets begin by rasterizing the portion of the circle satisfying $0 \le x \le y$ (shown in red). The general shell for an algorithm like the midpoint algorithm is

```
void rasterize_circle(int r)
{
  int j = first_y;
  for(int i = first_x, stop_criterion; i++)
  {
    draw(i,j);
    if(update_criterion)
      j++; // or j--;
  }
}
```



(a) What values should be used for `first_x` and `first_y`?

`first_x=0;` and `first_y=r;`

(b) Should we use `j++` or `j--`? Why?

`j--;` As we move right (`i++`;) the curve is decreasing, so `j` must decrease.

(c) Suggest something reasonable for `stop_criterion`.

`i <= j;` is a reasonable choice.

(d) For `update_criterion`, we need a function $g(x, y)$ such that $g(x, y) \le 0$ if and only if (x, y) is inside the circle. Suggest a suitable function $g(x, y)$. (You are not allowed to use square roots.)

$g(x, y) = x^2 + y^2 - r^2$

(e) For `update_criterion`, we want to test $g(i + a, j + b)$ at one point $(i + a, j + b)$. What should we use for the constants a and b?

$a = 1$, $b = -\frac{1}{2}$. Recall that we are testing the point halfway between $(i+1, j)$ and $(i+1, j-1)$.

(f) Finally, `update_criterion` will take the form $g(i+a, j+b) < 0$ or $g(i+a, j+b) > 0$. Which inequality do we want? You must justify your answer.

We want $g(i+1, j-\frac{1}{2}) > 0$. We must decrement `j` if the candidate point is too high, which places it outside the circle.

Problem 12 (3 points)

Clip the segment (in homogeneous coordinates) with endpoints $A = (-4, 1, 3, 5)$ and $B = (-2, 5, 3, 3)$.

Note that both vertices satisfy all of the clipping constraints except $y \leq w$, which B fails. Thus, we only need to clip against the plane $y = w$. Let

$$\begin{aligned}
P &= (1-\gamma)A + \gamma B \\
0 &= P_y - P_w \\
&= ((1-\gamma)A_y + \gamma B_y) - ((1-\gamma)A_w + \gamma B_w) \\
&= (1 - \gamma + 5\gamma) - (5(1-\gamma) + 3\gamma) \\
&= -4 + 6\gamma \\
\gamma &= \frac{2}{3} \\
P &= \frac{1}{3}A + \frac{2}{3}B \\
P &= \begin{pmatrix} -\frac{8}{3} \\ \frac{11}{3} \\ 3 \\ \frac{11}{3} \end{pmatrix}
\end{aligned}$$

Then, AP is the clipped segment.

Problem 13 (2 points)

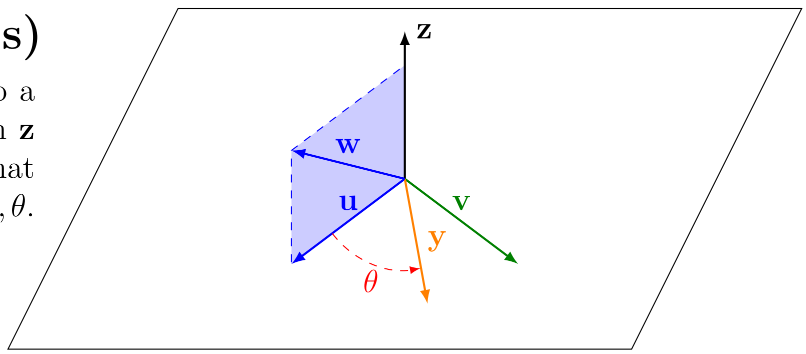
What is self-shadowing, and how do we prevent it?

Self-shadowing is when we cast a shadow ray to see if any object blocks the light and discover that the light is blocked by an intersection with ourself at distance zero. We prevent it by discarding intersections that are very close to zero.

In the following sequence of problems, we will work out formulas for 3D rotations about a given axis $\mathbf{z}$ by a given angle θ. You may assume $\|\mathbf{z}\| = 1$. We will call the rotation matrix $\mathbf{R}$; our goal is to derive a formula for $\mathbf{R}$. We will do this by considering its effects on an arbitrary vector $\mathbf{w}$. That is, we will work out a formula for $\mathbf{Rw}$ that depends only on $\mathbf{w}$, $\mathbf{z}$, and θ. *All of the parts below may be completed independently and in any order. You may assume the results of earlier problems when completing later ones. In particular, some parts ask you to compute new variables; you may use those variables in later parts, even if you have not computed them. It is important to read earlier problems, since they contain information you may need to complete later ones.*

Problem 14 (1 points)

We begin by decomposing $\mathbf{w}$ into a part $a\mathbf{z}$ along the axis of rotation $\mathbf{z}$ and a part $\mathbf{u}$ orthogonal to $\mathbf{z}$, so that $\mathbf{w} = \mathbf{u} + a\mathbf{z}$. Find a in terms of $\mathbf{w}, \mathbf{z}, \theta$. You may assume $\mathbf{u} \cdot \mathbf{z}$.



$\mathbf{w} = \mathbf{u} + a\mathbf{z}$. Using orthogonality and the fact that $\mathbf{z}$ is normalized, $\mathbf{w} \cdot \mathbf{z} = \mathbf{u} \cdot \mathbf{z} + a\mathbf{z} \cdot \mathbf{z} = a$.

Problem 15 (1 points)

Find $\mathbf{u}$ in terms of $\mathbf{w}, \mathbf{z}, \theta, a$.

$\mathbf{u} = \mathbf{w} - a\mathbf{z}$.

Problem 16 (1 points)

The plane shown has normal direction $\mathbf{z}$. Find another vector $\mathbf{v}$ in this plane, such that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{z} \cdot \mathbf{v} = 0$. Select your vector so that $\|\mathbf{u}\| = \|\mathbf{v}\|$. Your vector may depend on $\mathbf{w}, \mathbf{z}, \mathbf{u}, \theta, a$.

$\mathbf{v} = \mathbf{z} \times \mathbf{u}$.

Problem 17 (1 points)

Vectors in this plane will rotate by the full angle θ. Let $\mathbf{y} = \mathbf{R}\mathbf{u}$ be the vector that will be obtained by rotating $\mathbf{u}$ around the axis $\mathbf{z}$ by angle θ. Since this vector is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, we can write $\mathbf{y} = b\mathbf{u} + c\mathbf{v}$. Since rotations do not change lengths, we also know $\|\mathbf{y}\| = \|\mathbf{u}\|$. Find b in terms of θ. (Partial credit if you express in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \theta, a$, but if you have done this correctly, only θ will remain after simplification.)

$$
\begin{aligned}
\mathbf{y} &= b\mathbf{u} + c\mathbf{v} \\
\mathbf{y} \cdot \mathbf{u} &= b\mathbf{u} \cdot \mathbf{u} + c\mathbf{v} \cdot \mathbf{u} \\
\|\mathbf{y}\|\|\mathbf{u}\| \cos\theta &= b\|\mathbf{u}\|^2 \\
b &= \cos\theta
\end{aligned}
$$

Problem 18 (1 points)

Find c. There are two solutions; either is fine. Full credit if c is expressed in terms of b, θ only; partial credit if in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \theta, a, b$.

$$
\begin{aligned}
\mathbf{y} &= b\mathbf{u} + c\mathbf{v} \\
\mathbf{y} \cdot \mathbf{y} &= (b\mathbf{u} + c\mathbf{v}) \cdot (b\mathbf{u} + c\mathbf{v}) \\
\|\mathbf{y}\|^2 &= b^2\mathbf{u} \cdot \mathbf{u} + 2bc\mathbf{u} \cdot \mathbf{v} + c^2\mathbf{v} \cdot \mathbf{v} \\
\|\mathbf{y}\|^2 &= b^2\|\mathbf{u}\|^2 + c^2\|\mathbf{v}\|^2 \\
c^2 &= 1 - b^2 \\
c &= \pm\sqrt{1 - b^2} = \pm\sin\theta
\end{aligned}
$$

Problem 19 (1 points)

Let $\mathbf{q} = \mathbf{Rz}$. Find $\mathbf{q}$ in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{y}, \theta, a, b, c$.
Rotations do not change vectors along the axis of rotation, so $\mathbf{q} = \mathbf{z}$.

Problem 20 (1 points)

Find $\mathbf{Rw}$ in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{q}, \theta, a, b, c$.
$\mathbf{Rw} = \mathbf{Ru} + a\mathbf{Rz} = \mathbf{y} + a\mathbf{q}$

Problem 21 (1 points)

Find a matrix $\mathbf{M}$ such that $\mathbf{Ms} = (\mathbf{s} \cdot \mathbf{z})\mathbf{z}$ for any vector $\mathbf{s}$. Your matrix $\mathbf{M}$ may only depend on the vector $\mathbf{z}$.
$\mathbf{M} = \mathbf{z}\mathbf{z}^T$.

Problem 22 (1 points)

Find a matrix $\mathbf{A}$ such that $\mathbf{As} = \mathbf{z} \times \mathbf{s}$ for any vector $\mathbf{s}$. Your matrix $\mathbf{A}$ may only depend on the vector $\mathbf{z}$. (You will need to write out $\mathbf{A}$ in terms of the components of $\mathbf{z}$.)

$\mathbf{M} = \mathbf{z}\mathbf{z}^T$.

Problem 23 (3 points)

Construct a 3×3 matrix that performs each of the following 2D operations (in homogeneous coordinates). If no such matrix exists, explain why.

(a) $(x, y) \to (y, x)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) $(x, y) \to (1 + \frac{1}{x}, \frac{y}{x})$

$$(x, y, 1) \to (1 + x, y, x)$$
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(c) $(x, y) \to (\frac{1}{x}, \frac{1}{y})$

Clearing the fractions using homogeneous coordinates gives (y, x, xy), but xy cannot be achieved with a matrix. No such matrix exists.

(d) $(x, y) \to (\frac{y+1}{x+1}, 2)$

$$(x, y, 1) \to (y + 1, 2x + 2, x + 1)$$
$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

(e) $(x,y) \to (\frac{x}{x+y}, \frac{y}{x+y})$

$$(x,y,1) \to (x,y,x+y)$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

(f) $(x,y) \to (\frac{ax+by+c}{rx+sy+t}, \frac{dx+ey+f}{rx+sy+t})$, for given constants a,b,c,d,e,f,r,s,t.

$$(x,y,1) \to (ax+by+c, dx+ey+f, rx+sy+tx, y, x+y)$$
$$\begin{pmatrix} a & b & c \\ d & e & f \\ r & s & t \end{pmatrix}$$

Problem 24 (3 points)

Construct a 3×3 matrix that performs each of the following 2D operations (in homogeneous coordinates). You may express it as the product of other matrices if you prefer.

(a) Rotate 45° counterclockwise about the origin.

$$\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Translate 2 units in the positive x direction.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Rotate 90° counterclockwise about the point $(1, 2)$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 25 (2 points)

A raytracing scene has a yellow-colored sphere illuminated by a magenta-colored light. What color will the sphere appear in the image?

The sphere is color $(1, 1, 0)$ and the light is color $(1, 0, 1)$. These two vectors will be multiplied componentwise (whether ambient, specular, or diffuse), resulting in $(1, 0, 0)$, which is red. Depending on the lighting, this may be scaled down, resulting in a darker red. Note that a lighter red or pink is not possible.

Problem 26 (2 points)

We can clip a triangle against the sides of the image by simply not vising pixels outside the image while rasterizing. The z-buffer lets us clip based on the near and far planes. Nevertheless, we must still implement a separate clipping step. Why? (It is not just an optimization.)

Clipping needs to happen before the perspective divide, since otherwise objects that are outside the viewing area can become projected into the viewing area. Rasterization happens after the perspective divide.

Problem 27 (2 points)

The ray-object intersection problem often results in a polynomial that must be solved for t. In the ray-plane case, the polynomial had degree 1. In the ray-sphere case, the polynomial had degree 2. The ray-torus intersection also results in a polynomial in t that must be solved. What degree do you think this polynomial would have and why? (A torus is the shape of a doughnut. It is round and has a hole in the middle.) This question can be answered without doing any algebra. You don't even need to know what the equation for a torus is.

A ray can intersect a doughnut four times, so the polynomial that is solved must have four real roots. This implies that the polynomial must have degree *at least* four. In fact, the degree is exactly four.

[1]

[1] Total points: 60