CS 130, Final

Solutions

Read the entire exam before beginning. **Manage your time carefully.** This exam has 75 points; you need 55 to get full credit. Additional points are extra credit. 75 points → 2.4 min/point. 55 points → 3.3 min/point.

Problem 1 (2 points)

When constructing a bounding box hierarchy, the bounding boxes at the leaves are computed from the rendering primitives (spheres, triangles, etc.) Boxes at other nodes of the tree must be computed as the union of the boxes from their child nodes. Construct an algorithm to compute the bounding box that is the union of two given bounding boxes. You may assume that a bounding box is represented by its minimum and maximum corners $A = (a_x, a_y, a_z)$ and $B = (b_x, b_y, b_z)$.

Problem 2 (2 points)

Plants absorb a portion of the sunlight that they receive and use it to produce sugars in a process called photosynthesis. What colors of light do you think the plants are using?

Plants appear green because they reflect the green light that they receive. They absorb the colors that are not green. They use red/orange light and blue/violet light for photosynthesis. They do not use green or yellow.

Problem 3 (2 points)

What are homogeneous coordinates and why does the graphics pipeline use them?

These are coordinates with an extra component (x, y, z, w) that are related to regular coordinates by $(x/w, y/w, z/w)$. The extra coordinate makes it possible to represent translations and projections as a matrix.

Problem 4 (2 points)

When viewed from 20 cm away, the whiteboard appears white. When viewed from a distance of 20 m away, the whiteboard appears to be approximately the same color and brightness. In the second case, the observer is standing 100 times farther away. Why does the whiteboard not appear 10,000 times dimmer (essentially completely black)?

In any fixed area of your retina, the light reaching your eye from any patch of the board will indeed be reduced by a factor of 10,000. However, the area of the board that reaches that part of your retina is multiplied by a factor of 10,000. The result is that the brightness stays about the same. You do still receive 10,000 times less light since the board shines that light over an area of your retina 10,000 times smaller.

Problem 5 (2 points)

A mesh contains the seven vertices A–G and the six triangles (B, C, A), (C, D, B), (E, D, B), (F, D, C), (D, E, F), (G, E, F). Fix the orientations of the triangles so that they are consistent.

An example of a correctly orientated mesh is (B, C, A), (C, B, D), (E, D, B), (F, C, D), (D, E, F), (G, F, E).

Problem 6 (3 points)

For each of the following examples, indicate whether the light source is best described as A=ambient, D=directional, P=point, or S=spotlight. For each, fill your answer (A, D, P, or S) in the table. No explanation is required. Your solution must be in the table to count.

#	object	answer
(a)	flashlight	S
(b)	candle	P
(c)	twilight†	A
(d)	light bulb	P
(e)	moonlight	D
(f)	sunlight	D

†Twilight is the period of the day after sunset (or before sunrise) when the sun is not visible in the sky but it is also not dark.

Problem 7 (2 points)

Write a routine that rasterizes a filled circle. That is, it draws pixels that lie on or inside the circle but not pixels that lie outside it. Your routine should be written in C++-like syntax and use the signature `void fill_circle(int x, int y, int r);`, where (x, y) and r are the circle's center and radius. You may call the routine `void draw(int x,int y);` to set the pixel (x, y). You may assume that the circle lies entirely inside the image.

```
void fill_circle(int x, int y, int r)
{
  for(int i=-r;i<=r;i++)
    for(int j=-r;j<=r;j++)
      if(i*i+j*j<=r*r)
        draw(x+i,y+j);
}
```

Problem 8 (2 points)

On the project, we discarded intersections closer than `small_t` when computing the closest intersection. Why?

When casting rays from a surface, such as for reflection rays and shadow rays, it is important that the ray not be considered to intersect the surface it starts on. This is done by discarding intersections closer than a tolerance `small_t`.

The next five questions are intended to be solved in sequence. Some questions use numerical results from prior questions. If you did not solve one of the earlier problems, you may refer to the solution using a variable.

Problem 9 (2 points)

Imagine that we are ray tracing a simple scene with a camera at $(-1,2,0)$. We are rendering a pixel at location $(0,1,0)$. What are the endpoint and direction for the ray that we will cast?

The ray's endpoint is $(-1,2,0)$. The (un-normalized) direction is $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, which normalizes to the ray direction $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

Problem 10 (2 points)

The scene contains only one object, a plane passing through the origin with normal $\langle 0,1,0\rangle$. Find the ray-plane intersection location.

The intersection location is $(1,0,0)$. (At distance $t = 2\sqrt{2}$ along the ray.)

Problem 11 (2 points)

What is the surface normal at the intersection location?

It is just the plane's normal: $\langle 0,1,0\rangle$.

Problem 12 (2 points)

There is a point light located at $(3,3,1)$. Compute the endpoint and direction for the shadow ray that must be cast.

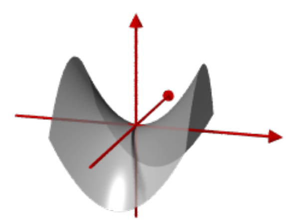
The endpoint is the intersection location $(1,0,0)$. The direction is the normalization of $\begin{pmatrix}3-1\\3-0\\1-0\end{pmatrix}$, which is $\frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix}$.

Problem 13 (2 points)

The plane is reflective, so we must cast a reflection ray. Compute the endpoint and direction for the reflection ray that must be cast.

The endpoint is the intersection location $(1,0,0)$. The direction is the reflected direction $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$.

The next three problems refer to the hyperbolic paraboloid defined by the implicit function $f(x,y,z) = x^2 - y^2 - z$. You may assume that $f(x,y,z) > 0$ corresponds to *outside* of the surface. The surface is illustrated at right. These problems also refer to the *ray* defined by $g(t) = \begin{pmatrix}2t-1\\-t-1\\-2t-1\end{pmatrix}$, where $t \geq 0$. All of these problems are independent (you can solve them in any order, even if you have not solved earlier ones).



Problem 14 (2 points)

What are the direction and endpoint of the ray?

Endpoint is $g(0) = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$. Direction is $\begin{pmatrix}\frac{2}{3}\\-\frac{1}{3}\\-\frac{2}{3}\end{pmatrix}$. (Don't forget to normalize!)

Problem 15 (2 points)

Compute the closest intersection *location* of the surface with the ray. [Hint: the answer should not contain square roots; if it does not, check your math.]

Plugging $g(t)$ into $f(x,y,z)$ for the cone we get $(2t-1)^2-(-t-1)^2-(-2t-1)=3t^2-4t+1=(3t-1)(t-1)$. Thus, $t=\frac{1}{3},1$. The first is closer, so the closest intersection location is $g(\frac{1}{3})=(-\frac{1}{3},-\frac{4}{3},-\frac{5}{3})$. The farther intersection location is at $g(1)=(1,-2,-3)$.

Problem 16 (2 points)

What is the normal direction at an arbitrary point (x,y,z) lying on the surface? Don't worry about whether the normal points inwards or outwards.

$$
\begin{aligned}
f &= x^2 - y^2 - z \\
\nabla f &= \begin{pmatrix} 2x \\ -2y \\ -1 \end{pmatrix} \\
\|\nabla f\| &= \sqrt{4x^2+4y^2+1} \\
n = \frac{\nabla f}{\|\nabla f\|} &= \frac{1}{\sqrt{4x^2+4y^2+1}} \begin{pmatrix} 2x \\ -2y \\ -1 \end{pmatrix}
\end{aligned}
$$

An alternative strategy is to parameterize the surface, such as with (x,y) and $\mathbf{w}=(x,y,z)=$

$(x, y, x^2 - y^2)$

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ x^2 - y^2 \end{pmatrix} \quad \mathbf{v}_x = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \quad \mathbf{v}_y = \begin{pmatrix} 0 \\ 1 \\ -2y \end{pmatrix} \quad \mathbf{v}_x \times \mathbf{v}_y = \begin{pmatrix} (0)(-2y) - (2x)(1) \\ (2x)(0) - (1)(-2y) \\ (1)(1) - (0)(0) \end{pmatrix} = \begin{pmatrix} -2x \\ 2y \\ 1 \end{pmatrix}$$

This agrees, up to sign, with what we obtained the original way.

Problem 17 (3 points)

Construct 3×3 transformation matrices that accomplish each of the following in 2D:

(a) Rotation by 90 degrees (you can rotate clockwise or counterclockwise).
(b) Translate in the y direction by 3.
(c) Scale in the y direction by 3.
(d) Translate in the y direction by 3, followed by a scale in the y direction by 3.
(e) Scale in the y direction by 3, followed by a translation in the y direction by 3.
(f) Swap the x and y components.

$$(a) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad (c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 9 \\ 0 & 0 & 1 \end{pmatrix} \qquad (e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad (f) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 18 (2 points)

The 4×4 transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

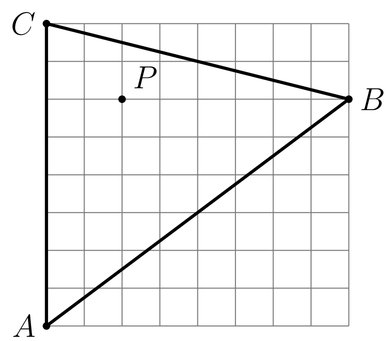
expresses a transformation from one 3D point (x, y, z) to another 3D point (x', y', z'). Find x', y', and z'.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 2y \\ 3z \\ x + 2y + 3z \end{pmatrix}$$

$$x' = \frac{x}{x + 2y + 3z} \qquad y' = \frac{2y}{x + 2y + 3z} \qquad z' = \frac{3z}{x + 2y + 3z}$$

Problem 19 (2 points)

Find the barycentric weights for the point P in the triangle below.



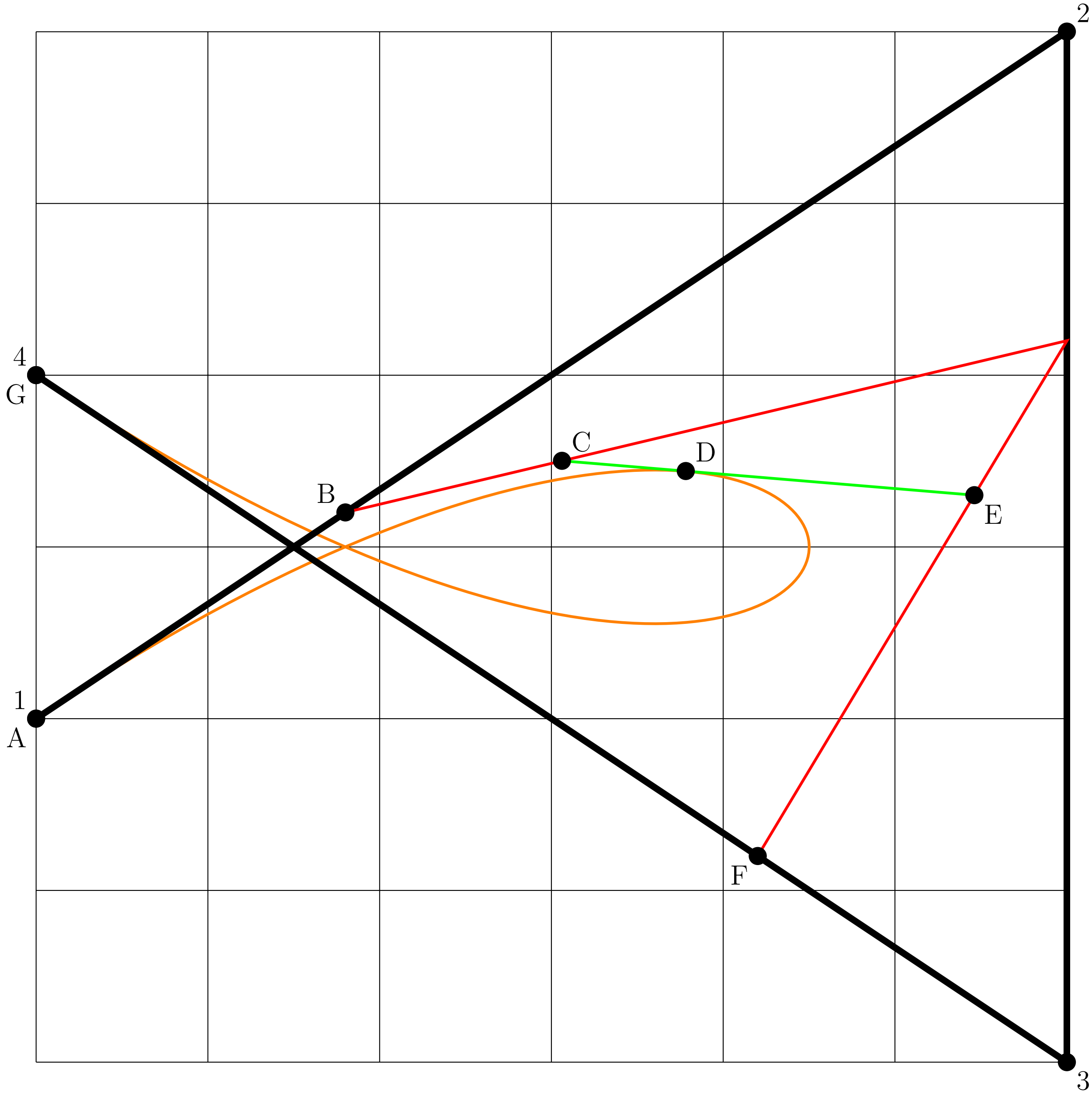
To do this, we need to compute the areas of the triangles, which we can then use to compute barycentric weights. Note that all of the triangles we need have an edge that is horizontal or vertical, so $A = \frac{1}{2}bh$ is easy to compute.

$$\text{area}(ABC) = 32 \qquad \text{area}(PBC) = 6 \qquad \text{area}(APC) = 8 \qquad \text{area}(ABP) = 18$$

$$\alpha = \frac{\text{area}(PBC)}{\text{area}(ABC)} = \frac{6}{32} \qquad \beta = \frac{\text{area}(APC)}{\text{area}(ABC)} = \frac{8}{32} \qquad \gamma = \frac{\text{area}(ABP)}{\text{area}(ABC)} = \frac{18}{32}$$

Problem 20 (3 points)

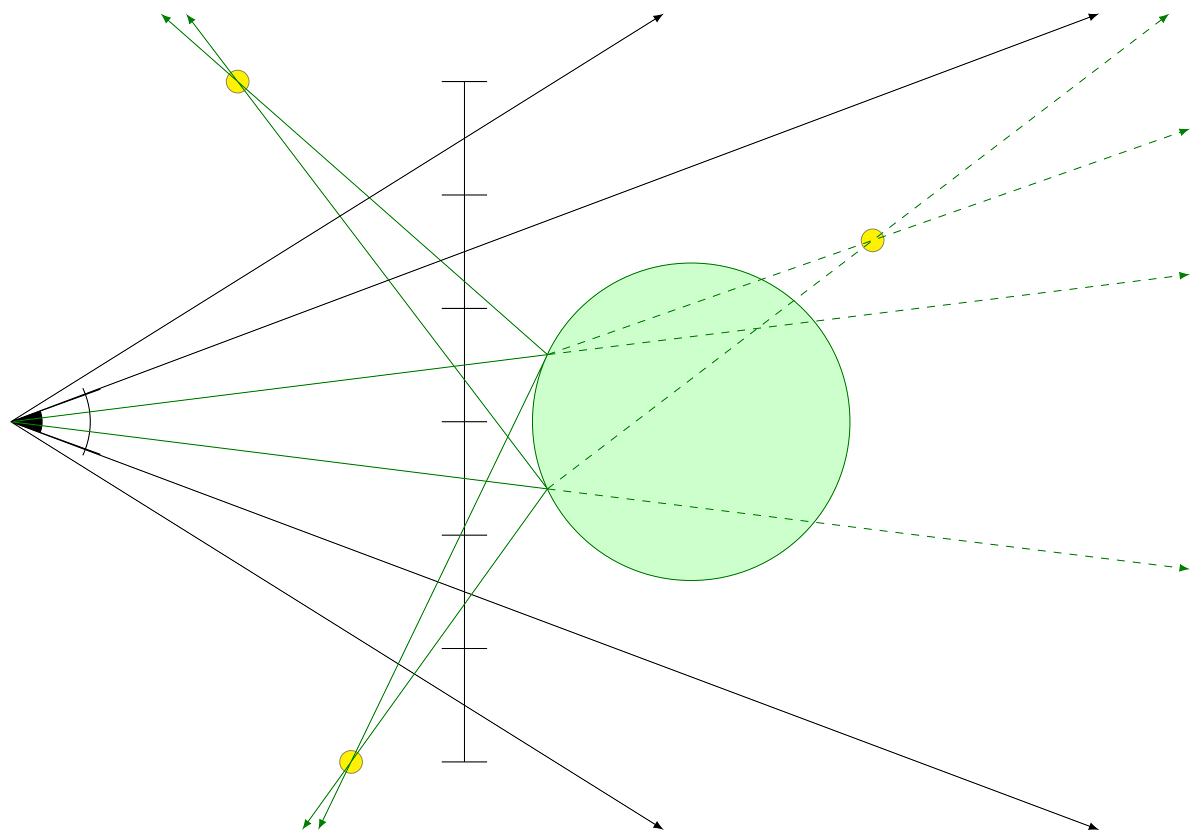
Geometrically subdivide the Bézier curve given by the control points below and at $t = 0.3$ along the curve. Label the new control points A, B, C, . . . , G. (The two resulting Bézier curves should share their common point, so only 7 control points are required.)



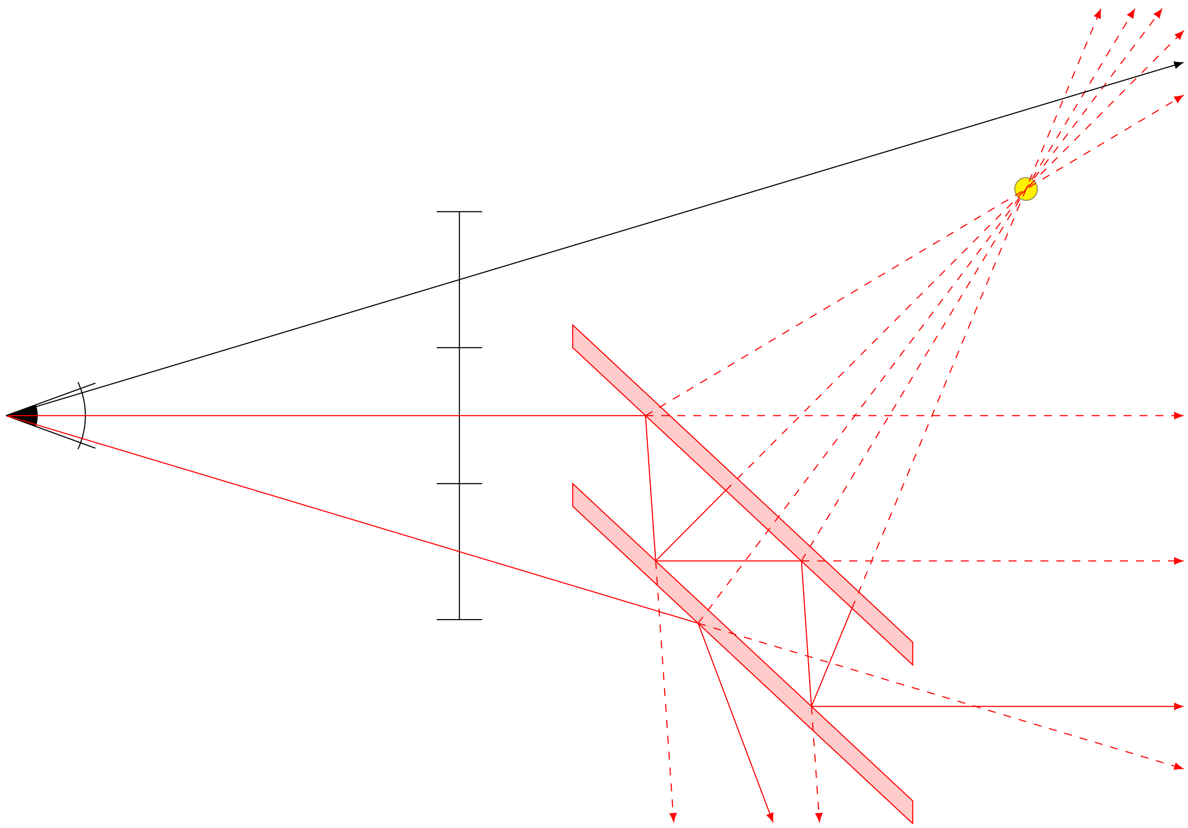
In the raytracing problems below, green objects are wood, red objects are reflective, and blue objects are transparent. The scenes are in 2D with a 1D image. yellow circles are point lights; the ray tracer supports shadows. Draw all of the rays that would be cast while raytracing each scene. Use a maximum recursion depth of 5. (Don't worry about precisely what counts as depth 5; I just care that recursion is being performed correctly when necessary and that important rays are not missing. There are no more than 20 rays in the "exact" solution.)

Problem 21 (4 points)

Note that this image has six pixels.



Problem 22 (4 points)



Problem 23 (2 points)

Clip the segment (in homogeneous coordinates) with endpoints $A = (-1, 0, 1, 2)$ and $B = (-2, -1, 0, 1)$.

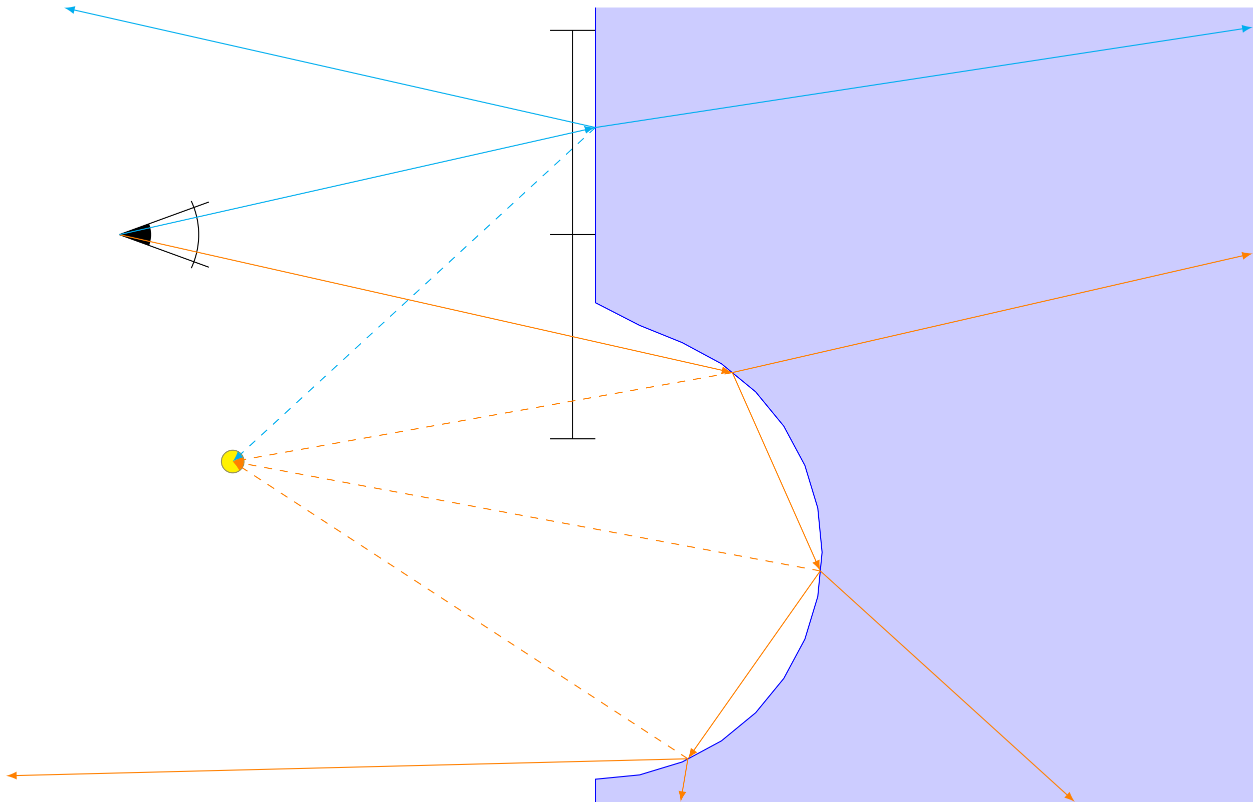
Note that both vertices satisfy all of the clipping constraints except $-w \leq x$, which B fails.

Thus, we only need to clip against the plane $x = -w$. Let

$$\begin{aligned}
P &= (1-\gamma)A + \gamma B \\
0 &= P_x + P_w \\
&= ((1-\gamma)A_x + \gamma B_x) + ((1-\gamma)A_w + \gamma B_w) \\
&= ((1-\gamma)(-1) + \gamma(-2)) + ((1-\gamma)(2) + \gamma(1)) \\
&= -1 + \gamma - 2\gamma + 2 - 2\gamma + \gamma \\
&= 1 - 2\gamma \\
\gamma &= \frac{1}{2} \\
P &= \frac{1}{2}A + \frac{1}{2}B \\
P &= \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}
\end{aligned}$$

Then, AP is the clipped segment.

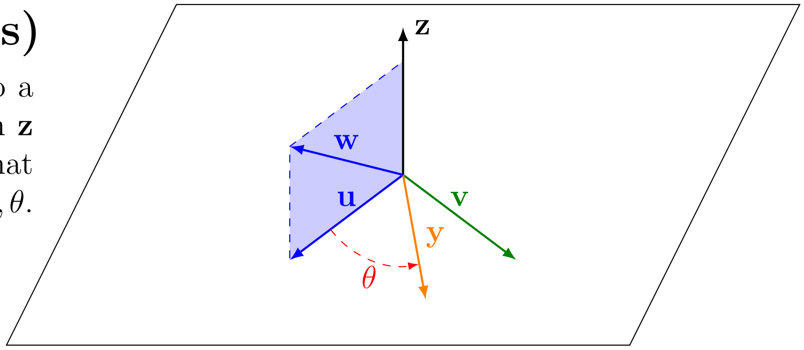
Problem 24 (4 points)



In the following sequence of problems, we will work out formulas for 3D rotations about a given axis $\mathbf{z}$ by a given angle θ. You may assume $\|\mathbf{z}\| = 1$. We will call the rotation matrix $\mathbf{R}$; our goal is to derive a formula for $\mathbf{R}$. We will do this by considering its effects on an arbitrary vector $\mathbf{w}$. That is, we will work out a formula for $\mathbf{Rw}$ that depends only on $\mathbf{w}$, $\mathbf{z}$, and θ. *All of the parts below may be completed independently and in any order. You may assume the results of earlier problems when completing later ones. In particular, some parts ask you to compute new variables; you may use those variables in later parts, even if you have not computed them. It is important to read earlier problems, since they contain information you may need to complete later ones.*

Problem 25 (2 points)

We begin by decomposing $\mathbf{w}$ into a part $a\mathbf{z}$ along the axis of rotation $\mathbf{z}$ and a part $\mathbf{u}$ orthogonal to $\mathbf{z}$, so that $\mathbf{w} = \mathbf{u} + a\mathbf{z}$. Find a in terms of $\mathbf{w}, \mathbf{z}, \theta$. You may assume $\mathbf{u} \cdot \mathbf{z}$.



$\mathbf{w} = \mathbf{u} + a\mathbf{z}$. Using orthogonality and the fact that $\mathbf{z}$ is normalized, $\mathbf{w} \cdot \mathbf{z} = \mathbf{u} \cdot \mathbf{z} + a\mathbf{z} \cdot \mathbf{z} = a$.

Problem 26 (2 points)

Find $\mathbf{u}$ in terms of $\mathbf{w}, \mathbf{z}, \theta, a$.

$\mathbf{u} = \mathbf{w} - a\mathbf{z}$.

Problem 27 (2 points)

The plane shown has normal direction $\mathbf{z}$. Find another vector $\mathbf{v}$ in this plane, such that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{z} \cdot \mathbf{v} = 0$. Select your vector so that $\|\mathbf{u}\| = \|\mathbf{v}\|$. Your vector may depend on $\mathbf{w}, \mathbf{z}, \mathbf{u}, \theta, a$.

$\mathbf{v} = \mathbf{z} \times \mathbf{u}$.

Problem 28 (2 points)

Vectors in this plane will rotate by the full angle θ. Let $\mathbf{y} = \mathbf{R}\mathbf{u}$ be the vector that will be obtained by rotating $\mathbf{u}$ around the axis $\mathbf{z}$ by angle θ. Since this vector is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, we can write $\mathbf{y} = b\mathbf{u} + c\mathbf{v}$. Since rotations do not change lengths, we also know $\|\mathbf{y}\| = \|\mathbf{u}\|$. Find b in terms of θ. (Partial credit if you express in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \theta, a$, but if you have done this correctly, only θ will remain after simplification.)

$$
\begin{aligned}
\mathbf{y} &= b\mathbf{u} + c\mathbf{v} \\
\mathbf{y} \cdot \mathbf{u} &= b\mathbf{u} \cdot \mathbf{u} + c\mathbf{v} \cdot \mathbf{u} \\
\|\mathbf{y}\|\|\mathbf{u}\| \cos\theta &= b\|\mathbf{u}\|^2 \\
b &= \cos\theta
\end{aligned}
$$

Problem 29 (2 points)

Find c. There are two solutions; either is fine. Full credit if c is expressed in terms of b, θ only; partial credit if in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \theta, a, b$.

$$
\begin{aligned}
\mathbf{y} &= b\mathbf{u} + c\mathbf{v} \\
\mathbf{y} \cdot \mathbf{y} &= (b\mathbf{u} + c\mathbf{v}) \cdot (b\mathbf{u} + c\mathbf{v}) \\
\|\mathbf{y}\|^2 &= b^2\mathbf{u} \cdot \mathbf{u} + 2bc\mathbf{u} \cdot \mathbf{v} + c^2\mathbf{v} \cdot \mathbf{v} \\
\|\mathbf{y}\|^2 &= b^2\|\mathbf{u}\|^2 + c^2\|\mathbf{v}\|^2 \\
c^2 &= 1 - b^2 \\
c &= \pm\sqrt{1 - b^2} = \pm\sin\theta
\end{aligned}
$$

Problem 30 (2 points)

Let $\mathbf{q} = \mathbf{R}\mathbf{z}$. Find $\mathbf{q}$ in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{y}, \theta, a, b, c$.
Rotations do not change vectors along the axis of rotation, so $\mathbf{q} = \mathbf{z}$.

Problem 31 (2 points)

Find $\mathbf{R}\mathbf{w}$ in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{q}, \theta, a, b, c$.
$\mathbf{R}\mathbf{w} = \mathbf{R}\mathbf{u} + a\mathbf{R}\mathbf{z} = \mathbf{y} + a\mathbf{q}$

Problem 32 (2 points)

Find a matrix $\mathbf{M}$ such that $\mathbf{M}\mathbf{s} = (\mathbf{s} \cdot \mathbf{z})\mathbf{z}$ for any vector $\mathbf{s}$. Your matrix $\mathbf{M}$ may only depend on the vector $\mathbf{z}$.
$\mathbf{M} = \mathbf{z}\mathbf{z}^T$.

Problem 33 (2 points)

Find a matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{s} = \mathbf{z} \times \mathbf{s}$ for any vector $\mathbf{s}$. Your matrix $\mathbf{A}$ may only depend on the vector $\mathbf{z}$. (You will need to write out $\mathbf{A}$ in terms of the components of $\mathbf{z}$.)

$\mathbf{M} = \mathbf{z}\mathbf{z}^T$.

[1]

[1] Total points: 75