For each question below, choose the correct answer from the provided list. Make sure to keep notes with your work, as you will be required to upload them at the end of this problem.

Q1.1

2 Points

Ariana has 3 milk bones and she wants to give them to three of her 6 dogs, with each getting one bone. In how many ways this can be done?

- ○ 120
- ◉ 20
- ○ 6
- ○ 27
- ○ 10
- ○ none of the above

EXPLANATION

Choose first dog in 6 ways, second in 5 ways and third in 4 ways, for the total of 120 choices of three dogs. But since the ordering does not matter, you need to divide by 6 (the number of permutations of the 3 dogs that receive a bone), so the answer is 20. Alternatively, use the formula for 3-subsets of 6 elements, $\binom{6}{3} = 20$.

Q1.2

2 Points

Ariana also has 3 other treats: Barkies, Poopsicles, and K9nies. She wants to distribute one piece of each among her 6 dogs. There are no restrictions on which or how many treats each dog may receive. In how many ways this can be done?

- ○ 36
- ○ 20
- ○ 18
- ○ 729
- ◉ 216
- ○ none of the above

EXPLANATION

Each treat can be assigned in 6 ways, so the number of combined choices is $6 \cdot 6 \cdot 6 = 216$. This is the number of functions from a 3-element set to a 6-element set.

Q1.3

2 Points

Bob wants to visit Lithuania, Latvia, Estonia, Ukraine, Poland and Slovakia during his visit to Europe. In how many orders he can visit these countries?

- ○ 360
- ○ 36
- ○ 6
- ◉ 720
- ○ 120
- ○ none of the above

EXPLANATION

This is the number of permutations of 6 elements, $6! = 720$.

Q1.4

2 Points

At a salsa lesson there are five boys: Adam, Bob, Chad, Dave, and Evan, and five girls: Fay, Gail, Helen, Iris, and Jane. In how many ways we can pair them into five pairs, with each pair having one boy and one girl?

- ○ 25
- ◉ 120
- ○ 625
- ○ 10
- ○ 5
- ○ none of the above

EXPLANATION

You can compute it as follows: Choose a partner for Adam in 5 ways, then there will be 4 choices for Bob's parter, then 3 choices for Chad's partner, and two choices for Dave's partner. Even will have no choice. So the total number of possibilities is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Alternatively, this can be thought of as ordering the girls in all possible ways, so we can use the formula for the number of permutations of 5 elements, $5! = 120$.

Q1.5 Upload notes with your work (required).

0 Points

No files uploaded

Q2

8 Points

Give the numerical value of each expression. Make sure to keep notes with your work, as you will be required to upload them at the end of this problem.

Q2.1

2 Points

$2 + 4 + 6 + ... + 30 =$

- ○ 60
- ○ 96
- ○ 480
- ○ 120
- ○ 160
- ◉ none of the above

EXPLANATION

Use the formula for the sum of the first n integers, $n(n+1)/2$. This gives $2 + 4 + 6 + ... + 30 = 2(1 + 2 + ... + 15) = 2 \cdot 15 \cdot 16/2 = 240$. Alternatively, you can use a formula for the sum of an arithmetic sequence: $n(a_1 + a_n)/2 = 15 \cdot (2 + 30)/2 = 240$.

Q2.2

2 Points

$\sum_{i=1}^{6} 3^i =$

- ○ 2187
- ○ 2186
- ◉ 1092
- ○ 1093
- ○ 729
- ○ none of the above

EXPLANATION

Use the formula for the sum of a geometric sequence, $\sum_{i=0}^{6} 3^i = (3^7 - 1)/(3 - 1) = 1093$. Then subtract $3^0 = 1$, because our summation starts with index 1 not 0, so the result is 1092. It can also be calculated as follows: $\sum_{i=1}^{6} 3^i = 3 \cdot \sum_{i=0}^{5} 3^i = 3 \cdot (3^6 - 1)/(3 - 1) = 3 \cdot 364 = 1092$.

Q2.3

2 Points

$\log_5 625^3 =$

- ◉ 12
- ○ 5
- ○ 25
- ○ 4
- ○ 10
- ○ none of the above

EXPLANATION

$\log_5 625^3 = 3 \cdot \log_5 625 = 3 \cdot 4 = 12.$

Q2.4

2 Points

$\binom{11}{7} =$

- ○ 108
- ○ 333
- ○ 1320
- ◉ 330
- ○ 110
- ○ none of the above

EXPLANATION

$\binom{11}{7} = \frac{11!}{7!4!} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2} = 330.$

Q2.5 Upload notes with your work (required).

0 Points

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Q3

8 Points

For each statement below determine whether it is true or false. In these questions, $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Make sure to keep notes with your work, as you will be required to upload them at the end of this problem.

Q3.1

2 Points

$\exists x \in \mathbb{Z}\,[\,x^4 - x^2 - 2 = 0\,]$

- [ ] True
- [x] False

EXPLANATION

The only candidates for integral roots are $1, 2, -1, -2$, and neither works. Alternatively, you can substitute $y = x^2$, which gives equation $y^2 - y - 2 = 0$. This equation does not have integer roots, and thus the original one doesn't either.

Q3.2

2 Points

$\forall x \in \mathbb{R}\ [\, x^2 + 10 > 6x\,]$

- [x] True
- [ ] False

EXPLANATION

For all $x \in \mathbb{R}$ we have $x^2 - 6x + 10 > 0$, because the polynomial $x^2 - 6x + 10$ does not have real roots (as its discriminant is negative) and the coefficient of x^2 is positive.

Q3.3

2 Points

$\forall x \in \mathbb{R}\ [\, (x^2+1)^{-x^2} \leq 1\,]$

- [x] True
- [ ] False

EXPLANATION

We have $x^2+1\ge 1$ and the exponent $-x^2$ is never positive.

Q3.4

2 Points

$\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}\ [\,(x^2 + y^2 + 1)^{x+y+1} = 1\,]$

- (selected) True
- False

EXPLANATION

For any y, choose $x = -y - 1$, and then $x^2 + y^2 + 1 > 0$ and the exponent $x + y + 1$ is 0 .

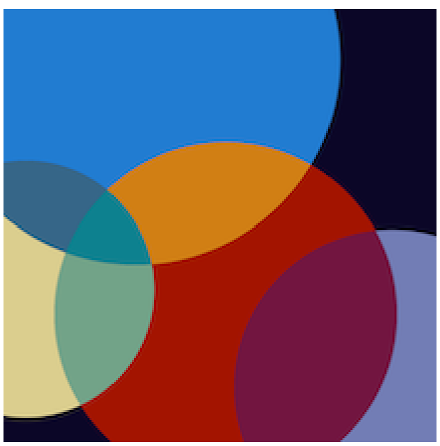
Q3.5 Upload notes with your work (required).

0 Points

No files uploaded

Q4 Sets and Relations

8 Points



Make sure to keep notes with your work, as you will be required to upload them at the end of this problem.

Q4.1

2 Points

Let $A = \{a, b\}$, $B = \{a, b, c\}$, $C = \{a, c, d\}$, and $D = \{\{a, b\}, c\}$. Which of the relations below are true? (There may be multiple true answers.)

- [x] $A \subseteq B$
- [ ] $C \subseteq A \cup B$
- [x] $B \setminus C \subseteq A$
- [x] $A \cap D = \emptyset$
- [x] $A \in D$
- [ ] $A \subseteq D$

Q4.2
2 Points

Let A, B, and C be sets. Which of the statements below are always true? Recall that $\emptyset$ represents the empty set. Notation $X \not\subseteq Y$ means that X is not a subset of Y.

- [x] If $A \subseteq B$ then $A \subseteq B \cup C$.
- [ ] If $A \subseteq B$ then $A \subseteq B \cap C$.
- [x] $A \cap B \subseteq A \cup B$.
- [x] If $A \cap B = \emptyset$ and $A \cap C = \emptyset$ then $A \cap (B \cup C) = \emptyset$.
- [ ] If $A \not\subseteq B$ and $A \not\subseteq C$ then $A \not\subseteq B \cup C$.

Q4.3
2 Points

Let Q be a relation on real numbers defined by xQy iff $|x - y| \leq 1$. Determine whether relation Q has the following properties:

- [x] Reflexive
- [x] Symmetric
- [ ] Transitive
- [ ] Equivalence

EXPLANATION

Q is reflexive because $|x-x|=0 \leq 1$. It is symmetric because $|x-y| \leq 1$ implies $|y-x| \leq 1$. It is not transitive. For example, $1Q2$ and $2Q3$ but $\neg 1Q3$. Therefore Q is not an equivalence relation.

Q4.4

2 Points

Let R be the relation defined by the table below. In this table "1" represents "true" and "0" represents "false".

R	a	b	c	d	e
a	1	0	1	0	0
b	0	1	0	1	1
c	1	0	1	0	0
d	0	1	0	1	1
e	0	1	0	1	1

Determine whether relation R has the following properties:

- [x] Reflexive
- [x] Symmetric
- [x] Transitive
- [x] Equivalence

EXPLANATION

It's reflexive because all entries on the diagonal are 1's. It is also symmetric, because its matrix (table) is symmetric with respect to the diagonal. The transitivity can be checked by exhausting all cases. (There aren't many cases. It is sufficient to check if $xRy \wedge yRz \Rightarrow xRz$ only for $x \neq y$ and $y \neq z$.) Another way to do this is to swap rows $2, 3$ and columns $2, 3$. This shows that a, c are mutually related but not related to b, d, e, and b, d, e are mutually related, but not related to a, c. Therefore R is an equivalence relation. Its equivalence classes are $\{a, c\}$ and $\{b, d, e\}$.

Q4.5 Upload notes with your work (required).

0 Points

No files uploaded

Q5 Number Theory

8 Points

Make sure to keep notes with your work, as you will be required to upload them at the end of this problem.

Q5.1

2 Points

The prime factors (without repetitions) of 17325 are

- ○ $2, 5, 7$
- ○ $3, 7, 11$
- ○ $5, 5, 7$
- ○ $3, 7, 11$
- ◉ $3, 5, 7, 11$
- ○ none of the above

EXPLANATION

The factorization of 17325 is $3^2 \cdot 5^2 \cdot 7^1 \cdot 11^1$.

Q5.2

2 Points

Let $x = \gcd(3^3 4^1 5^1 7^9, 2^{10} 5^7 9^1 13^{12})$. Determine the value of x.

- ○ 5
- ◉ 180
- ○ 20
- ○ 60
- ○ 220
- ○ none of the above

EXPLANATION

For every prime p, the factorization of $\gcd(x, y)$ contains the highest power of p that appears in the factorizations of both x and y. The first number's factorization is $2^2 3^3 5^1 7^9$ and the second number's factorization is $2^{10} 3^2 5^7 13^{12}$. So the factorization of x is $2^2 3^2 5^1$, which gives us that $x = 180$.

Q5.3
2 Points

Let $a = 64 \cdot 9 \cdot 121 \cdot 125 \cdot 17$ and $b = 4 \cdot 243 \cdot 11 \cdot 5 \cdot 169$. Determine the value of $\gcd(a, b)$.

- ○ 2970
- ◉ 1980
- ○ 330
- ○ 119
- ○ 2310
- ○ none of the above

EXPLANATION

The common prime powers in the factorizations of a and b are $2^2 = 4$, $3^2 = 9$, $5^1 = 5$ and $11^1 = 11$. So $\gcd(a, b) = 4 \cdot 9 \cdot 5 \cdot 11 = 1980$.

Q5.4
2 Points

Let a, b, c be positive natural numbers. Determine whether the following statement is true or false: If $u \geq x$ and $v \geq y$ then $\gcd(u, v) \geq \gcd(x, y)$.

- ○ True
- ◉ False

EXPLANATION

To show that it's false, it's sufficient to give a counter-example. As one example, take $u = 3$, $v = 5$, and $x = 2$, $y = 2$. Then $\gcd(u, v) = 1$, but $\gcd(x, y) = 2$.

Q5.5 Upload notes with your work (required).
0 Points

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Q6
12 Points

Make sure to keep notes with your work, as you will be required to upload them at the end of this problem.

Q6.1
3 Points

Prove that equation $x^2 + 2x + 1 + y^2 = 0$ has only one solution: $x = -1$ and $y = 0$. Individual steps of a correct proof are given below, you only need to choose the correct ordering of these steps.

Proof:
(1) In the second case, when $y \neq 0$, rewrite this equation as $x^2 + 2x + 1 + y^2 = (x + 1)^2 + y^2$.
(2) Summarizing, we showed that in the first case the only solution is $x = -1$ and $y = 0$, and in the second case there is no solution --- completing the proof.
(3) We will consider two cases: when $y = 0$ and when $y \neq 0$.
(4) Since $(x+1)^2 \geq 0$ and $y^2 > 0$, we obtain that $x^2 + 2x + 1 + y^2 = (x+1)^2 + y^2 > 0$, that is the equation does not have a solution in this case.
(5) In the first case, for $y = 0$, this is a quadratic equation $x^2 + 2x + 1 = 0$ and its only root is $x = -1$.
QED

From the list below, select an ordering of these statements that will form a correct proof:

- ○ 1-3-4-5-2
- ◉ 3-5-1-4-2
- ○ 4-1-5-2-3
- ○ 3-2-5-4-1
- ○ 3-1-5-4-2

Q6.2

3 Points

Consider the following statement: *"We have a group of people consisting of 6 Ukrainians, 5 Poles, and 7 Slovaks. Some people in the group greet each other with a handshake (they shake hands only once). Prove that if 110 handshakes were exchanged in total, then two people of the same nationality shook hands"*.

The proof below contains some missing phrases. From the lists below, choose correct phrases to form a complete and correct proof.

Proof: We will estimate the maximum number of handshakes between people of different nationality. The number of handshakes between Ukrainians and Poles *(Phrase 1)*. The number of handshakes between Ukrainians and Slovaks *(Phrase 2)*. The number of handshakes between Poles and Slovaks *(Phrase 3)*. Thus the total number of handshakes between people of different nationalities *(Phrase 4)*. Since the total number of handshakes is 110, and *(Phrase 4)*, two people of the same nationality must have shaken hands. **QED**

Choose a correct Phrase 1:

- ○ is at most $\binom{5}{2} = 10$
- ○ is at least 5
- ○ is at most $6^2 = 36$
- ○ equals $6 + 5 = 11$
- ◉ is at most $6 \cdot 5 = 30$

Choose a correct Phrase 2:

- ○ equals $6+7=13$
- ○ is at most $\binom{6}{2}=15$
- ○ is at least 7
- ◉ is at most $6\cdot 7=42$
- ○ is at least 6

Choose a correct Phrase 3:

- ◉ is at most $5\cdot 7=35$
- ○ is at most $\binom{7}{2}=21$
- ○ is at least 7
- ○ is at least 6
- ○ equals $5+7=12$

Choose a correct Phrase 4:

- ◉ cannot exceed $30+42+35=107$
- ○ is at most $6\cdot 5\cdot 7=210$
- ○ is at least $6+5+7=18$
- ○ equals $10+15+21=46$
- ○ is at most $11+13+12=36$

Choose a correct Phrase 4:

- ○ $110<210$
- ◉ $107<110$
- ○ $110>37$
- ○ $37>36$
- ○ $210>107$

Q6.3
3 Points

For integers x, y, notation $x|y$ means that x is a divisor of y. (For example $5|30$, but it is not true that $5|21$.) $\mathbb{N}$ denotes the set of natural numbers. (Recall that in this class we assume that $0 \in \mathbb{N}$). Consider the following claim:

Claim: $\forall n \in \mathbb{N}\ 3|(2^{2n} - 1)$.

The inductive proof of this claim is given below, but in the inductive part the steps of the argument are out of order. Give a correct ordering of these steps. (The argument must be correct mathematically and grammatically as well.)

Proof: We apply mathematical induction. In the base case, when $n = 0$, we have $2^{2n} - 1 = 2^0 - 1 = 0$, and 0 is a multiple of 3 (because $0 = 3 \cdot 0$), so the claim holds for $n = 0$.

In the inductive step, let k be some arbitrary positive integer and assume that the claim holds for $n = k$, that is $3|(2^{2k} - 1)$. We now proceed as follows.

(1) This implies that $2^{2(k+1)} - 1$ is a multiple of 3, completing the inductive step.
(2) $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4 \cdot 2^{2k} - 1 = 4(3c + 1) - 1 = 12c + 3 = 3(4c + 1)$.
(3) From the inductive assumption, we have that $2^{2k} - 1 = 3c$, for some integer c.
(4) Next, we compute $2^{2n} - 1$ for the next value of n, that is for $n = k + 1$:
(5) This gives us that $2^{2k} = 3c + 1$.

We proved the base case, for $n = 0$, and that the claim holding for $n = k$ implies that it holds for $n = k + 1$. By the principle of induction, this proves the claim for all n. **QED**

Choose the correct ordering in the derivation above:

- ○ 1-2-3-4-5
- ○ 2-5-4-3-1
- ○ 3-2-1-4-5
- ◉ 3-5-4-2-1

Q6.4
3 Points

Below you are given an attempted proof of the following statement: *"Equation $x(x^4 + x^2 - x + 1) = -2$ does not have any solutions in real numbers"*. Find all errors in this proof by selecting steps that are not valid (that is, they do not follow from the assumptions or from earlier steps).

Proof: Assume that x is an integer that satisfies $x(x^4 + x^2 - x + 1) = -2$.

(1) Rearranging this equation, we obtain $x^5 + x^3 - x^2 + x + 2 = 0$.
(2) The polynomial in this equation can be factored as follows: $x^5 + x^3 - x^2 + x + 2 = (x^3 - x^2 + 1)(x^2 + x + 2)$.
(3) This implies that $x^3 - x^2 + 1 = 0$ and $x^2 + x + 2 = 0$.
(4) The second equation in step (3) is a quadratic equation with negative discriminant.
(5) This implies that no real number x satisfies the second equation in step (3).
(6) Therefore equation $x(x^4 + x^2 - x + 1) = -2$ does not have any solutions in real numbers.

Select the step or steps in this argument that are invalid:

- [ ] (1)
- [ ] (2)
- [x] (3)
- [ ] (4)
- [ ] (5)
- [ ] (6)

EXPLANATION

Step (3) is wrong, because $(x^3 - x^2 + 1)(x^2 + x + 2) = 0$ only implies that at least one of expressions $x^3 - x^2 + 1$ and $x^2 + x + 2$ is 0, not necessarily both.

Q6.5 Upload notes with your work (required).

0 Points

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