CS 130, Final

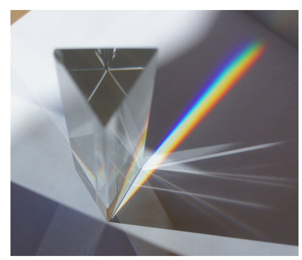
Solutions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

Read the entire exam before beginning. **Manage your time carefully.** This exam has 100 points; you need 80 to get full credit. Additional points are extra credit. 100 points → 1.8 min/point. 80 points → 2.25 min/point.

Problem 1 (2 points)



A prism is a transparent object that has different indexes of refraction for different wavelengths of light. As a result, for example, red and yellow light bend by different angles as they pass through the prism. In this way, sunlight is divided into a rainbow upon passing through a prism. (See the photograph to the right.) What would be observed if light from a computer screen displaying a solid white background were passed through the prism instead of white light from the sun?

Instead of a rainbow, you might see colored bands for red, green, and blue (in the same places that they occur in the rainbow) with no light between those bands. If the red, green, and blue are not pure, a few bands are observed. A rainbow will not be observed. (For example, sunlight contains light that is actually yellow. Yellow in a computer monitor is red plus green.)

Problem 2 (2 points)

We can clip a triangle against the sides of the image by simply not visiting pixels outside the image while rasterizing. The z-buffer lets us clip based on the near and far planes. Nevertheless, we must still implement a separate clipping step. Why? (It is not just an optimization.)

Clipping needs to happen before the perspective divide, since otherwise objects that are outside the viewing area can become projected into the viewing area. Rasterization happens after the perspective divide.

Problem 3 (2 points)

What is an area light, and how can this effect be achieved by a raytracer?

Area lights are lights that occupy a finite area (as opposed to a point light). They are ray traced by casting rays to randomly sampled locations on the light to determine visibility. If a subset of rays see the light, the light will be partially shadowed.

Problem 4 (2 points)

The ray-object intersection problem often results in a polynomial that must be solved for t. In the ray-plane case, the polynomial had degree 1. In the ray-sphere case, the polynomial had degree 2. The ray-torus intersection also results in a polynomial in t that must be solved. What degree do you think this polynomial would have and why? (A torus is the shape of a doughnut. It is round and has a hole in the middle.) *This question can be answered without doing any calculations. You don't even need to know what the equation for a torus is.*

A ray can intersect a doughnut four times, so the polynomial that is solved must have four real roots. This implies that the polynomial must have degree *at least* four. In fact, the degree is exactly four.

Problem 5 (2 points)

What is the *difference* between C^1 and G^1 continuity in the context of a car traveling down a road?

Both the G^1 car and the C^1 car are traveling down a smooth road. The speed of the C^1 driver is continuous, but the speed of the G^1 car is discontinuous.

Problem 6 (2 points)

Let's say we were to implement texture mapping in our ray tracers. To do this, we create a new `Texture_Shader` class derived from `Shader`, which implements the `Shade_Surface` function. This function calculates a color from an image stored in the class. Once we have the texture color, we have some options. (1) Return the texture color. (2) Use the texture color as the diffuse color and do Phong shading. Option (2) works well, but option (1) does not. Explain why.

Option (1) is basically like flat shading (and if the texture is a constant color, it is exactly flat shading). There is no lighting, no shadows, etc. There will be no 3D look to objects at all. Option (2) fixes this.

Problem 7 (2 points)

What are homogeneous coordinates and why does the graphics pipeline use them?

These are coordinates with an extra component (x, y, z, w) that are related to regular coordinates by $(x/w, y/w, z/w)$. The extra coordinate makes it possible to represent translations and projections as a matrix.

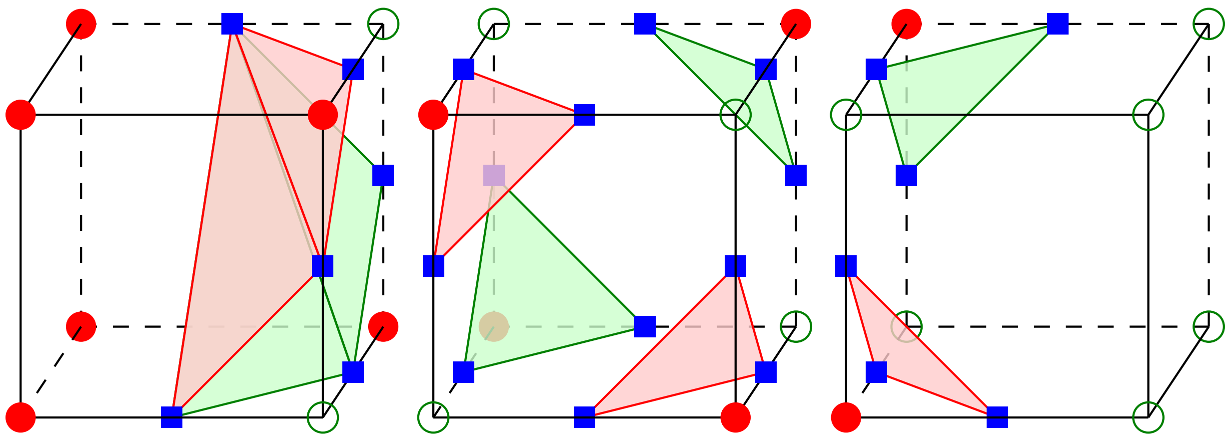
Problem 8 (3 points)

The graphics pipeline contains many stages, including the following: (a) perspective divide (divide by w), (b) geometry shader, (c) vertex shader, (d) rasterization, (e) z-buffering, (f) fragment shader, (g) clipping, and (h) tessellation shader. List these stages in the order that they are performed in the pipeline. If two steps may be performed in either order, you may select an order arbitrarily. No explanation is required.

Order: c, h, b, g, a, d, f, e. Under some circumstances, f and e can be reversed as an optimization.

Problem 9 (4 points)

Construct a consistent marching cubes triangulation of the cubes shown below. The cubes should actually be touching, but they have been separated apart for clarity. Draw squares at the vertices of your triangles.



Problem 10 (5 points)

A surface (rather like a spiral staircase) is given by the parametric equations $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$. What is the normal direction at an arbitrary point on the surface? You may express your result as a function of (r, θ) or (x, y, z) as you prefer.

$$\mathbf{w} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \theta \end{pmatrix} \qquad \mathbf{w}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad \mathbf{w}_\theta = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 1 \end{pmatrix}$$

$$\mathbf{w}_r \times \mathbf{w}_\theta = \begin{pmatrix} (\sin\theta)1 - 0(r\cos\theta) \\ 0(-r\sin\theta) - (\cos\theta)1 \\ (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ r \end{pmatrix}$$

$$\|\mathbf{w}_r \times \mathbf{w}_\theta\|^2 = (\sin\theta)^2 + (-\cos\theta)^2 + r^2 = r^2 + 1$$

$$n = \frac{\mathbf{w}_r \times \mathbf{w}_\theta}{\|\mathbf{w}_r \times \mathbf{w}_\theta\|} = \frac{1}{\sqrt{r^2+1}} \begin{pmatrix} \sin\theta \\ -\cos\theta \\ r \end{pmatrix} = \frac{1}{\sqrt{x^2+y^2+1}\sqrt{x^2+y^2}} \begin{pmatrix} y \\ -x \\ x^2+y^2 \end{pmatrix}$$

Problem 11 (4 points)

Find the intersection between a ray (endpoint $\mathbf{p}$, direction $\mathbf{u}$) and a plane (point $\mathbf{z}$, normal $\mathbf{n}$). You may assume that there is exactly one intersection.

$$\begin{aligned}
\mathbf{x} &= \mathbf{p} + t\mathbf{u} \\
(\mathbf{x} - \mathbf{z}) \cdot \mathbf{n} &= 0 \\
(\mathbf{p} + t\mathbf{u} - \mathbf{z}) \cdot \mathbf{n} &= 0 \\
t(\mathbf{u} \cdot \mathbf{n}) &= (\mathbf{z} - \mathbf{p}) \cdot \mathbf{n} \\
t &= \frac{(\mathbf{z} - \mathbf{p}) \cdot \mathbf{n}}{\mathbf{u} \cdot \mathbf{n}} \\
\mathbf{x} &= \mathbf{p} + \frac{(\mathbf{z} - \mathbf{p}) \cdot \mathbf{n}}{\mathbf{u} \cdot \mathbf{n}}\mathbf{u}
\end{aligned}$$

Problem 12 (4 points)

Construct a transformation that maps numbers x in the interval $[a, b]$ to numbers y in the interval $[c, d]$. (All points x must map to a valid y, and all values y must be achieved by some value of x.)

$$\begin{aligned}
y &= rx + s \\
c &= ra + s \\
d &= rb + s \\
d - c &= r(b - a) \\
r &= \frac{d - c}{b - a} \\
s &= c - ra = c - \frac{d - c}{b - a}a \\
y &= rx + s \\
y &= \frac{d - c}{b - a}x + c - \frac{d - c}{b - a}a \\
y &= \frac{d - c}{b - a}(x - a) + c
\end{aligned}$$

Problem 13 (4 points)

Suggest formulas for *barycentric coordinates* for a point P relative to a tetrahedron with vertices A, B, C, and D. That is, construct formulas for weights α, β, γ, and δ such that $\alpha + \beta + \gamma + \delta = 1$ and $P = \alpha A + \beta B + \gamma C + \delta D$. You do not need to show that these properties are satisfied.

$$\alpha = \frac{\text{volume}(PBCD)}{\text{volume}(ABCD)} \qquad \beta = \frac{\text{volume}(APCD)}{\text{volume}(ABCD)} \qquad \gamma = \frac{\text{volume}(ABPD)}{\text{volume}(ABCD)} \qquad \delta = \frac{\text{volume}(ABCP)}{\text{volume}(ABCD)}$$

Problem 14 (2 points)

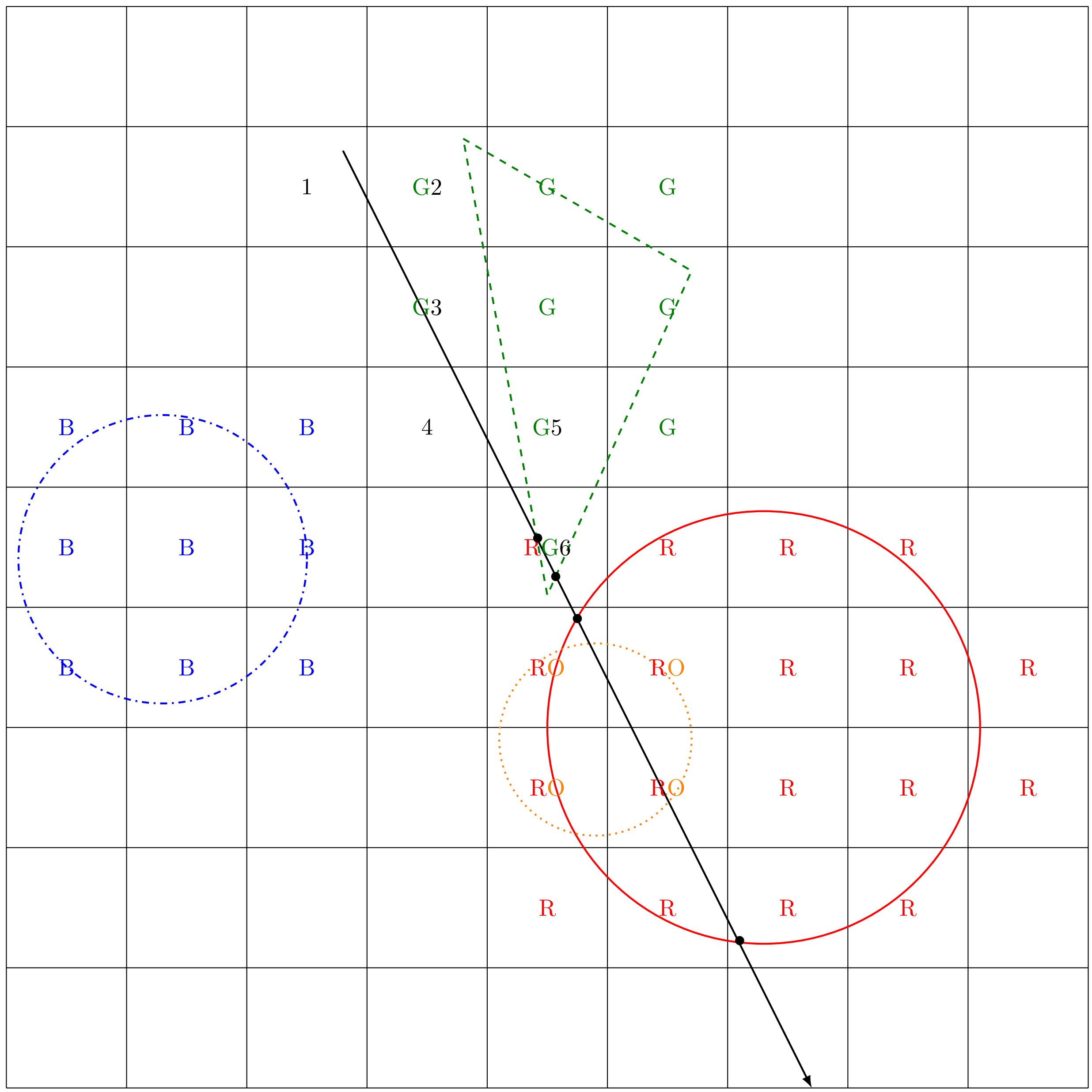
Assuming that you have α, β, γ, and δ from the previous problem, suggest an algorithm for deciding whether P is inside our outside of tetrahedron $ABCD$.

The point is inside if $\alpha, \beta, \gamma, \delta \geq 0$. It is outside otherwise.

Problem 15 (6 points)

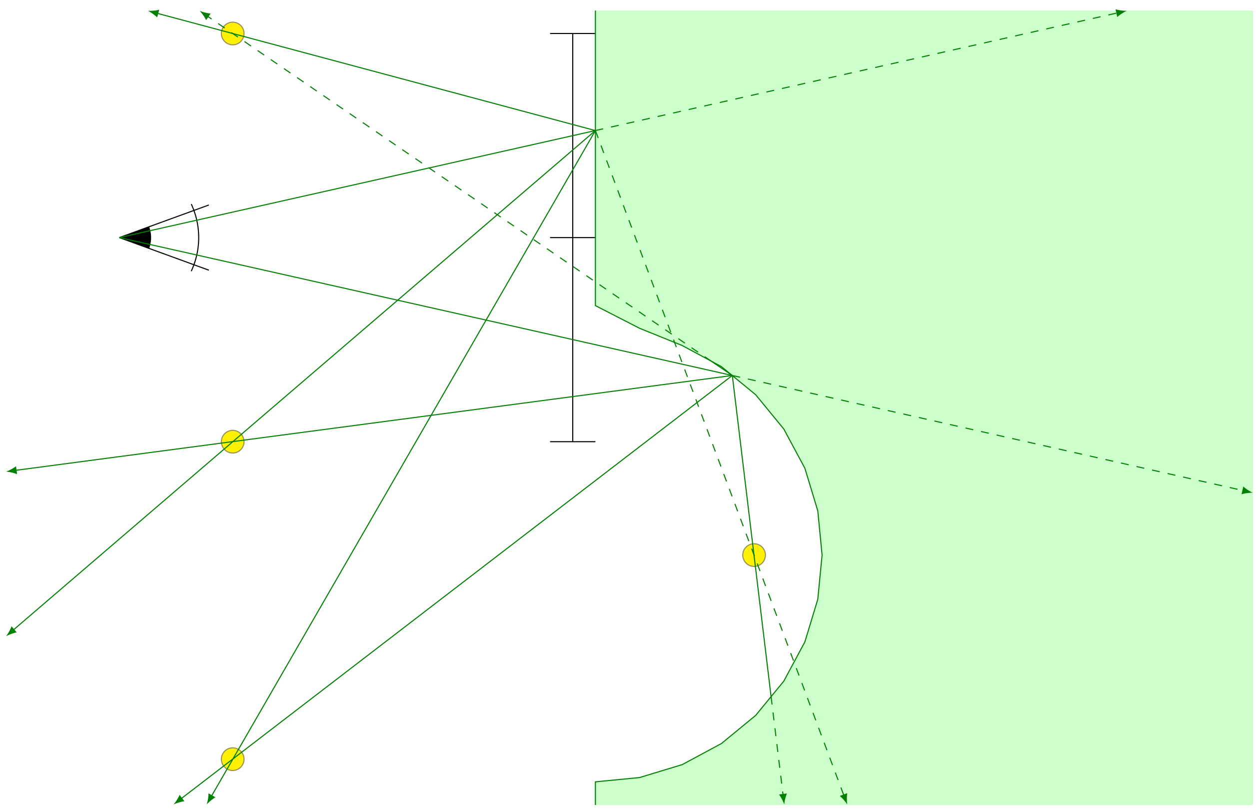
Below is a raytracing acceleration structure. Label the following:

1. Label grid cells with "R", "G", "B", or "O" to indicate that a pointer to the Red (———), Green (- - - - -), Blue (-·-·-·-), or Orange (..........) object will be stored there.
2. Number cells ("1", "2", "3", . . .) in the order they will be visited to test for intersections along the black ray. Cells that should not be visited should not be numbered.
3. Place a "•" at each intersection point that will be computed. Intersections that should not be computed should not be marked.



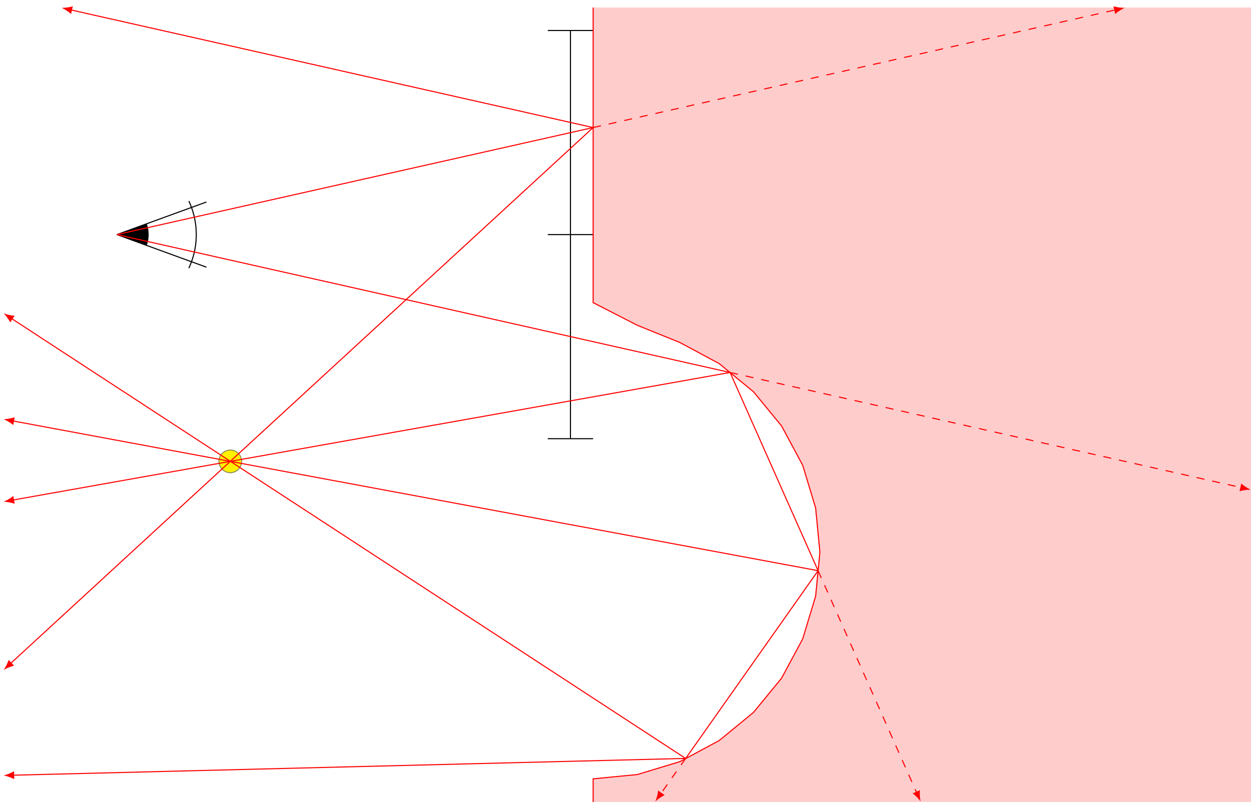
Problem 16 (10 points)

The object in the raytracing problem below is made of **wood**. The scene is in 2D with a 1D image. Each image has two pixels unless otherwise stated. yellow circles are point lights; the ray tracer supports shadows. Draw all of the rays that would be cast while raytracing this scene.



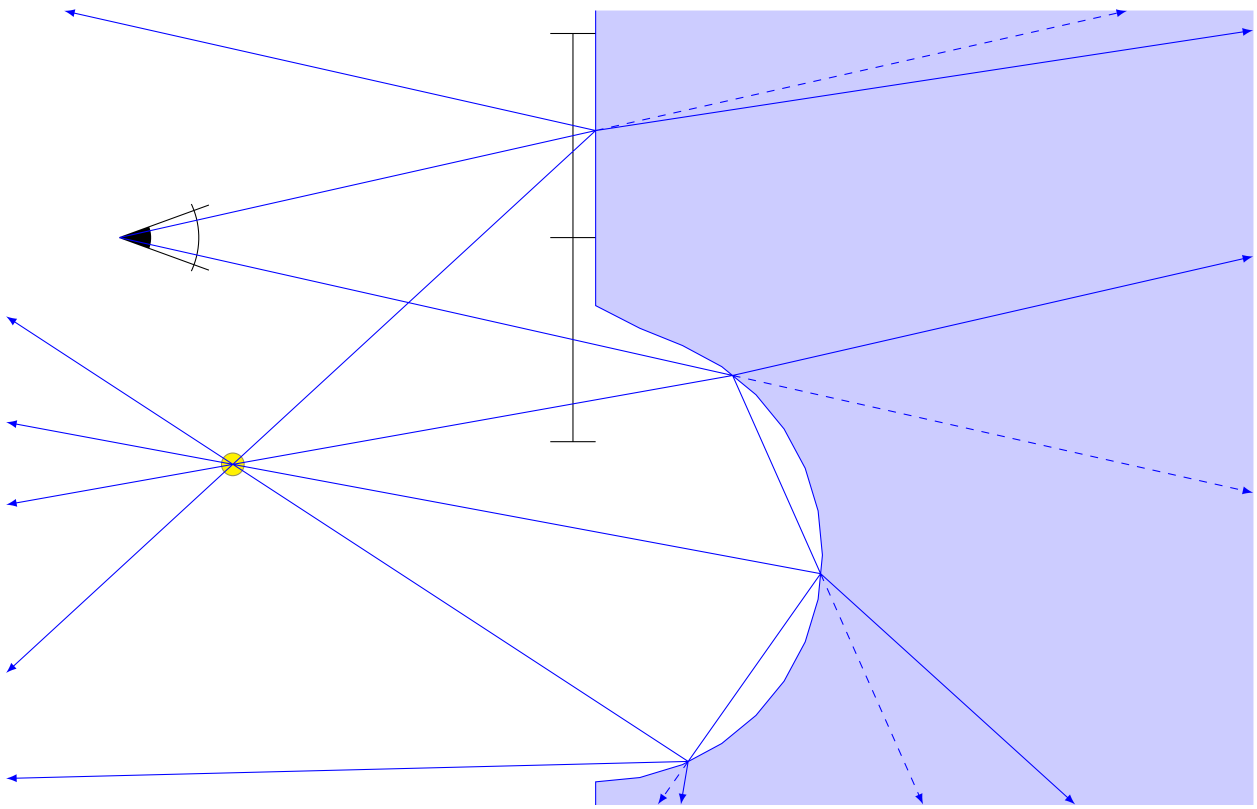
Problem 17 (10 points)

The object in the raytracing problem below is **reflective**. The scene is in 2D with a 1D image. Each image has two pixels unless otherwise stated. yellow circles are point lights; the ray tracer supports shadows. Draw all of the rays that would be cast while raytracing this scene.

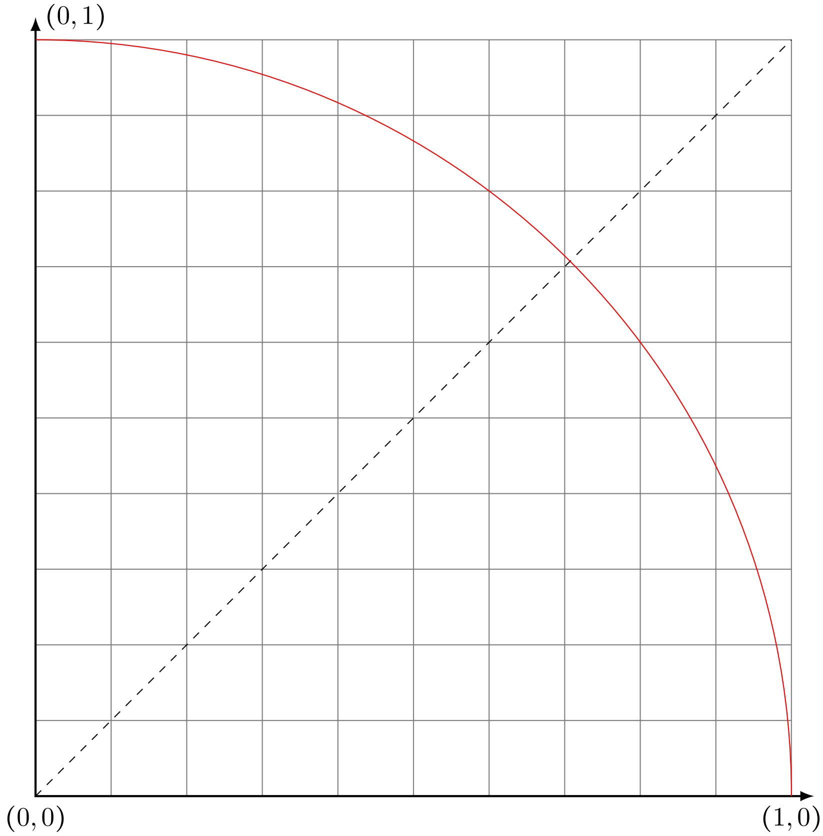


Problem 18 (10 points)

The object in the raytracing problem below is **transparent**. The scene is in 2D with a 1D image. Each image has two pixels unless otherwise stated. yellow circles are point lights; the ray tracer supports shadows. Draw all of the rays that would be cast while raytracing this scene.



In **Problems 19-23**, we will be approximating a quarter of the unit circle (shown below) using a cubic Bézier curve. Let $A = (a_0, a_1)$, $B = (b_0, b_1)$, $C = (c_0, c_1)$, and $D = (d_0, d_1)$ be the control points for the Bézier curve $P(t)$. In the problems that follow, we will calculate the eight degrees of freedom $a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1$. In many contexts (such as computer fonts), Bézier curves are used to approximate circles. This works well because the approximation is extremely good; the circle and its Bézier approximation in the figure would differ by less than the width of a human hair, far less than the width of the red line used to draw it.



Problem 19 (2 points)

What is $P(t)$ explicitly in terms of A, B, C, and D?

$P(t) = (1-t)^3 A + 3t(1-t)^2 B + 3t^2(1-t)C + t^3 D.$

Problem 20 (2 points)

(a) Let's begin by forcing the curve to agree with the quarter-circle at the endpoints by requiring $P(0) = (1, 0)$ and $P(1) = (0, 1)$. Use this to eliminate as many of the components of the control points as possible. This should eliminate four of the eight components. Hint: this step should not be tedious.

$P(0) = A$, so $a_0 = 1$ and $a_1 = 0$. $P(1) = D$, so $d_0 = 0$ and $d_1 = 1$.

Problem 21 (2 points)

(a) Next, lets force the curve to have the same slope as the quarter-circle at the endpoints. Use this to eliminate as many of the components of the control points as possible. This should eliminate two more of the components. Hint: this step should not be tedious.

The slope at $P(0)$ is along the vector from A to B. Since this should be vertical, $b_0 = a_0 = 1$. Similarly, $P(1)$ is along the vector from D to C. Since this should be horizontal, $c_1 = d_1 = 1$.

Problem 22 (2 points)

(a) Next, we will require the curve to be symmetric about $x = y$ (shown dashed in the figure above). In particular, if $P(t) = (x, y)$ then $P(1 - t) = (y, x)$. This should eliminate one more degree of freedom. Hint: this step should not be tedious.

The transformation $t \to 1 - t$ is equivalent to reversing the order of the control points. A and D already have the required symmetry. To make B and C have the symmetry, we need $c_0 = b_1$.

Problem 23 (2 points)

We are now left with just one degree of freedom, so we will need one additional constraint to solve for it. There are many ways to choose it. *(For example, we could make the area under the curve be $\frac{\pi}{4}$. We could minimize the maximum deviation of the curve from the circle. Forcing $P(\frac{1}{3})$ to lie on the unit circle also produces a very good approximation.)* In this problem, we will do something that is not quite as good but much simpler. We will force $P(\frac{1}{2})$ to lie on the unit circle. That is, $\|P(\frac{1}{2})\|^2 = 1$. Use this to solve for the final degree of freedom. Hint: this step is straightforward but slightly tedious. You should get a quadratic equation in one variable. You will need to do the previous problems in order to solve this part.

Let $(x, y) = P(\frac{1}{2})$. From the symmetry imposed in the previous problem, we know that the $x = y$. Thus, we only need to worry about the x component. $\|P(\frac{1}{2})\|^2 = 1$ reduces to $2x^2 = 1$.

$$\begin{aligned}
(x, y) = P(t) &= (1-t)^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3t(1-t)^2 \begin{pmatrix} 1 \\ b \end{pmatrix} + 3t^2(1-t) \begin{pmatrix} b \\ 1 \end{pmatrix} + t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
x &= \frac{1}{8} + \frac{3}{8} + \frac{3}{8}b = \frac{(4+3b)}{8} \\
2x^2 &= 1 \\
2\left(\frac{(4+3b)}{8}\right)^2 &= 1 \\
\frac{(4+3b)}{8} &= \pm\frac{1}{\sqrt{2}} \\
4 + 3b &= \pm 4\sqrt{2} \\
b &= \frac{-4 \pm 4\sqrt{2}}{3} \\
b &= \frac{-4 + 4\sqrt{2}}{3} \qquad \text{Note: need } b > 0
\end{aligned}$$

Problems 24-30 refer to objects F and G. The object F is represented implicitly by the function $f(\mathbf{x})$. ($f(\mathbf{x}) < 0$ is inside, $f(\mathbf{x}) > 0$ is outside, $f(\mathbf{x}) = 0$ is on the boundary.) Similarly, the object G is represented implicitly by the function $g(\mathbf{x})$. ($g(\mathbf{x}) < 0$ is inside, $g(\mathbf{x}) > 0$ is outside, $g(\mathbf{x}) = 0$ is on the boundary.) You may assume that $f(\mathbf{x})$ and $g(\mathbf{x})$ are continuous. Implicit representations that you construct should follow the same conventions. You may ignore degeneracies and pathological cases. You may use the solution of earlier problems to solve later problems, even if you have not solved the earlier problems. Hint: all of these have very simple solutions.

Problem 24 (2 points)

Let E be the *complement* of F. That is, something is in E if and only if it is outside F. Construct a function $e(\mathbf{x})$ that implicitly represents this object E.

$e(\mathbf{x}) = -f(\mathbf{x})$.

Problem 25 (2 points)

Let $H = F \cup G$ be the *union* of the objects. Construct a function $h(\mathbf{x})$ that implicitly represents this object H.

$h(\mathbf{x}) = \min(f(\mathbf{x}), g(\mathbf{x}))$.

Problem 26 (2 points)

Let $K = F \cap G$ be the *intersection* of the objects. Construct a function $k(\mathbf{x})$ that implicitly represents this object K.

$k(\mathbf{x}) = \max(f(\mathbf{x}), g(\mathbf{x}))$.

Problem 27 (2 points)

Let $M = F - G$ be the *difference* of the objects. Construct a function $m(\mathbf{x})$ that implicitly represents this object M.

$m(\mathbf{x}) = \max(f(\mathbf{x}), -g(\mathbf{x}))$. Note that this is just combining intersection and complement.

Problem 28 (2 points)

Let N be the object obtained by scaling F by a factor of 2 about the origin. Construct a function $n(\mathbf{x})$ that implicitly represents this object N.

$n(\mathbf{x}) = f(\mathbf{x}/2)$. Note that the 2 should be divided, not multiplied. One way to see that is to note that if the point $\mathbf{x}$ is on the surface of N then $\frac{\mathbf{x}}{2}$ is on the surface of F.

Problem 29 (2 points)

Let P be the *symmetric difference* of the objects F and G. (A point is in P if it is in F or G but not both.) Construct a function $p(\mathbf{x})$ that implicitly represents this object P.

$p(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$. Another way to construct a solution is to use $P = (F - G) \cup (G - F)$.

Problem 30 (2 points)

Let Q be the object represented by the implicit function $q(\mathbf{x}) = 2f(\mathbf{x})$. How is object Q related to object F?

The since $q(\mathbf{x})$ and $f(\mathbf{x})$ always agree on sign, the objects Q and F are the same.