CS 130, Final

Solutions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	$\sum$

Read the entire exam before beginning. **Manage your time carefully.** This exam has 60 points; you need 50 to get full credit. Additional points are extra credit. 60 points → 3.0 min/point. 50 points → 3.6 min/point.

Problem 1 (2 points)

Given two vectors $\vec{u}$ and $\vec{w}$, how do we determine whether the vectors are orthogonal?

$\vec{u} \cdot \vec{w} = 0$

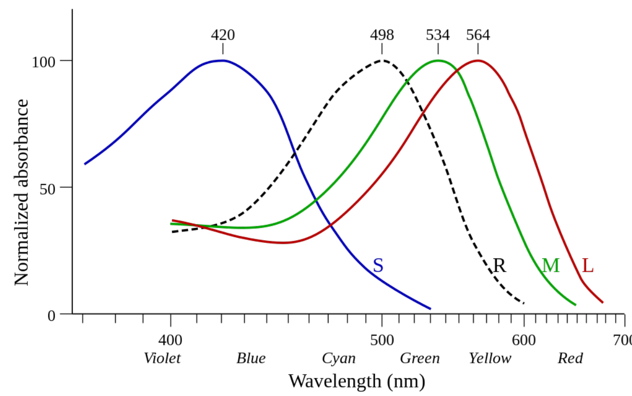
Problem 2 (2 points)

Given two vectors $\vec{u}$ and $\vec{w}$, how do we determine whether the vectors are parallel? You may assume 3D.

$\vec{u} \times \vec{w} = \vec{0}$. Another solution would be to check, for example, $\frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{w}}{\|\vec{w}\|}$. $\vec{u} \cdot \vec{w} = \|\vec{u}\|\|\vec{w}\|$ is another solution.

Problem 3 (4 points)

Humans have three types of cone cells, which have their peak sensitivities at around 564 nm (yellow), 534 nm (green), and 420 nm (blue), as shown in the figure. This allows us to see colors from about 700 nm (red) to 380 nm (violet)



Source: https://commons.wikimedia.org/wiki/File:Cone-absorbance-en.svg

(a) It is known that we can see and distinguish all (sufficiently different) wavelengths of light within this range as different colors. How do we know that this is true?

(b) In addition to the three types of cone cells, we also have rods, which peak at 498 nm, as shown in the figure. We only use these for vision in dim light, not for color vision in bright light. Let's suppose there was someone who was able to use rods to supplement their color vision. We will call this hypothetical person *Rodney*. How would the range of colors that Rodney can see differ from our own?

(c) What might Rodney be able to do that the rest of us cannot?

(d) Why would Rodney not be very impressed with color photography?

(a) There are many ways that we can see this. For example, if we look at a rainbow or the output of a prism, then we do not see gaps or repeated colors in the spectrum.

(b) It would not change. The sensitivity of the rod cells does not extend beyond the sensitivities of the cone cells, so no new frequencies would be visible.

(c) Rodney would be able to distinguish yellow from red+green, for example.

(d) Color photography relies on being able to faithfully represent a wide range of colors using RGB. This trick would not work on Rodney, so photos would look wildly different from the original.

Problem 4 (3 points)

Construct a 3×3 matrix that performs each of the following 2D operations (in homogeneous coordinates). You may express it as the product of other matrices if you prefer.

(a) Rotate 45° counterclockwise about the origin.

$$\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Translate 2 units in the positive x direction.

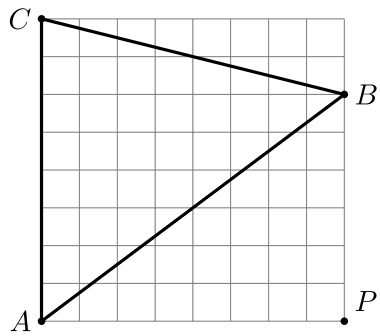
$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Rotate 90° counterclockwise about the point $(1, 2)$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 5 (3 points)

Find the barycentric weights for the point P in the triangle below.

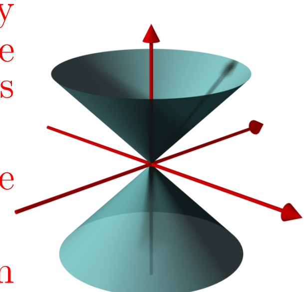


To do this, we need to compute the *signed* areas of the triangles, which we can then use to compute barycentric weights. Note that all of the triangles we need have an edge that is horizontal or vertical, so $A = \frac{1}{2}bh$ is easy to compute. We must also be careful about the orientations of the triangles. Triangles that are clockwise should have negative area.

$$\text{area}(ABC) = 32 \qquad \text{area}(PBC) = 24 \qquad \text{area}(APC) = 32 \qquad \text{area}(ABP) = -24$$

$$\alpha = \frac{\text{area}(PBC)}{\text{area}(ABC)} = \frac{24}{32} = \frac{3}{4} \qquad \beta = \frac{\text{area}(APC)}{\text{area}(ABC)} = \frac{32}{32} = 1 \qquad \gamma = \frac{\text{area}(ABP)}{\text{area}(ABC)} = \frac{-24}{32} = -\frac{3}{4}$$

The next few problems refer to the cone defined by the implicit function $f(x,y,z) = x^2 + y^2 - z^2$. You may assume that $f(x,y,z) > 0$ corresponds to *outside* of the cone. The cone is illustrated at right. These problems also refer to the *ray* defined by $g(t) = \begin{pmatrix} t+1 \\ t \\ t+2 \end{pmatrix}$, where $t \geq 0$. All of these problems are independent (you can solve them in any order, even if you have not solved earlier ones).



Problem 6 (2 points)

What are the direction and endpoint of the ray?

Endpoint is $g(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Direction is $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. (Don't forget to normalize!)

Problem 7 (4 points)

Compute all intersection *locations* of the cone with the ray. [Hint: the answer should come out nicely; if it does not, check your math.]

Plugging $g(t)$ into $f(x, y, z)$ for the cone we get $(t+1)^2 + t^2 - (t+2)^2 = t^2 - 2t - 3 = (t-3)(t+1)$. Thus, $t = -1, 3$. Since this is a ray and $t \geq 0$, we can exclude the first. This leaves $t = 3$, which corresponds to $g(3) = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$.

Problem 8 (4 points)

What is the normal direction at an arbitrary point (x, y, z) lying on the surface of the cone? Don't worry about whether the normal points inwards or outwards.

$$
\begin{aligned}
f &= x^2 + y^2 - z^2 \\
\nabla f &= \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix} \\
\|\nabla f\| &= 2\sqrt{x^2 + y^2 + z^2} \\
n = \frac{\nabla f}{\|\nabla f\|} &= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ -z \end{pmatrix}
\end{aligned}
$$

Note that $x^2 + y^2 - z^2 = 0$ so that $\sqrt{x^2 + y^2 + z^2} = \sqrt{2z^2} = |z|\sqrt{2}$.

An alternative strategy is to parameterize the surface, such as with (r, θ) and $\mathbf{w} = (x, y, z) = (r\cos\theta, r\sin\theta, r)$.

$$\begin{aligned}
\mathbf{w} &= \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix} \\
\mathbf{w}_r &= \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} \\
\mathbf{w}_\theta &= \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} \\
\mathbf{w}_r \times \mathbf{w}_\theta &= \begin{pmatrix} (\sin\theta)0 - 1(r\cos\theta) \\ 1(-r\sin\theta) - (\cos\theta)0 \\ (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) \end{pmatrix} = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ z \end{pmatrix} \\
\|\mathbf{w}_r \times \mathbf{w}_\theta\|^2 &= (-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2 = 2r^2 = 2z^2 \\
n &= \frac{\mathbf{w}_r \times \mathbf{w}_\theta}{\|\mathbf{w}_r \times \mathbf{w}_\theta\|} = -\frac{1}{|z|\sqrt{2}} \begin{pmatrix} x \\ y \\ -z \end{pmatrix}
\end{aligned}$$

Problem 9 (2 points)

The ray-object intersection problem often results in a polynomial that must be solved for t. In the ray-plane case, the polynomial had degree 1. In the ray-sphere case, the polynomial had degree 2. The ray-torus intersection also results in a polynomial in t that must be solved. What degree do you think this polynomial would have and why? (A torus is the shape of a doughnut. It is round and has a hole in the middle.) This question can be answered without doing any algebra. You don't even need to know what the equation for a torus is.

A ray can intersect a doughnut four times, so the polynomial that is solved must have four real roots. This implies that the polynomial must have degree *at least* four. In fact, the degree is exactly four.

Problem 10 (2 points)

When looking out a window during the day, we see whatever is outside. When we look out a window at night, we see ourselves. Why?

Transparent objects both reflect and transmit light. During the day, the light from the bright outdoors dominates, and we mostly just see what is outside. At night, there is little light coming from outside, so the reflected light dominates.

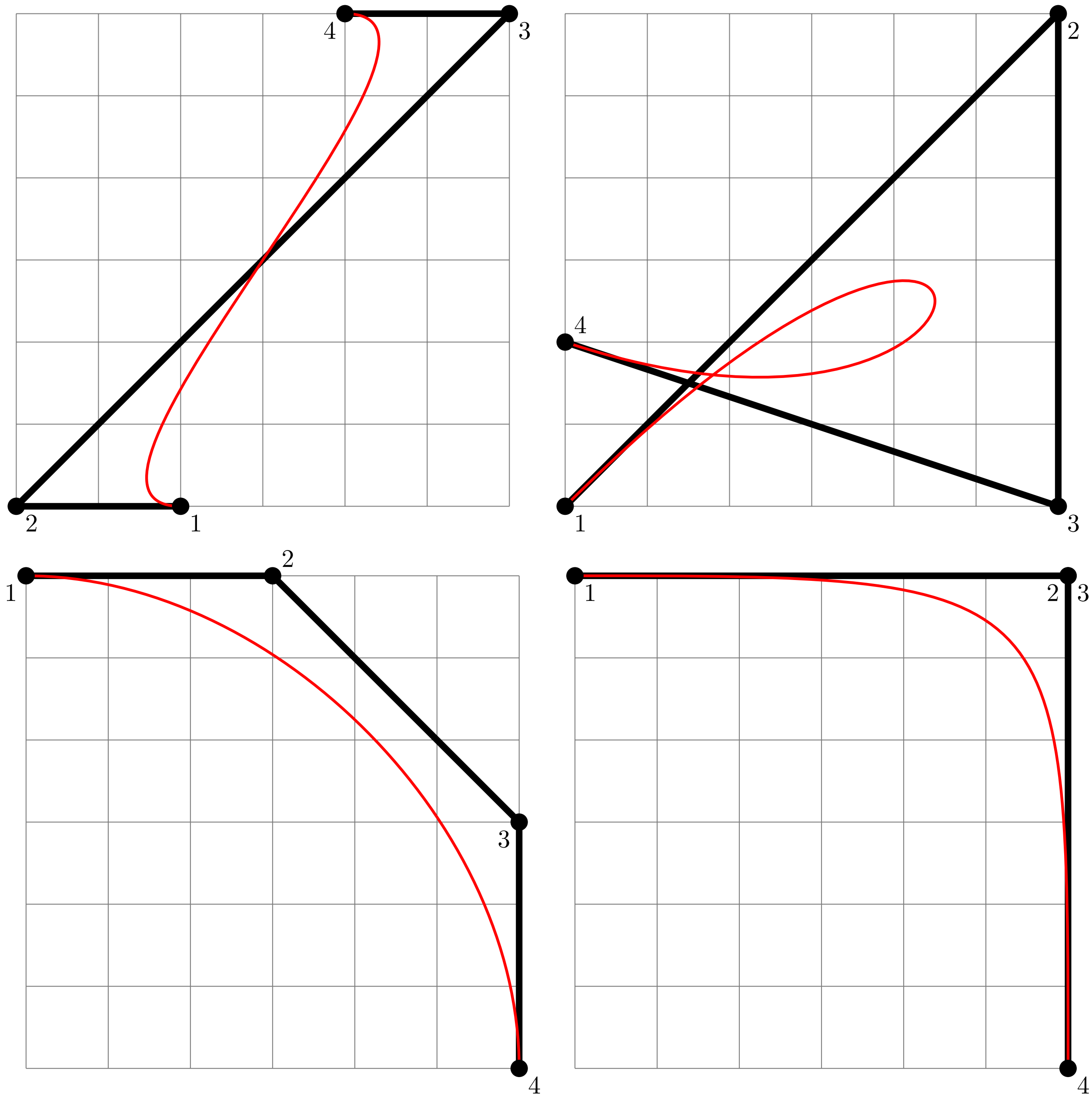
Problem 11 (2 points)

We can clip a triangle against the sides of the image by simply not vising pixels outside the image while rasterizing. The z-buffer lets us clip based on the near and far planes. Nevertheless, we must still implement a separate clipping step. Why? (It is not just an optimization.)

Clipping needs to happen before the perspective divide, since otherwise objects that are outside the viewing area can become projected into the viewing area. Rasterization happens after the perspective divide.

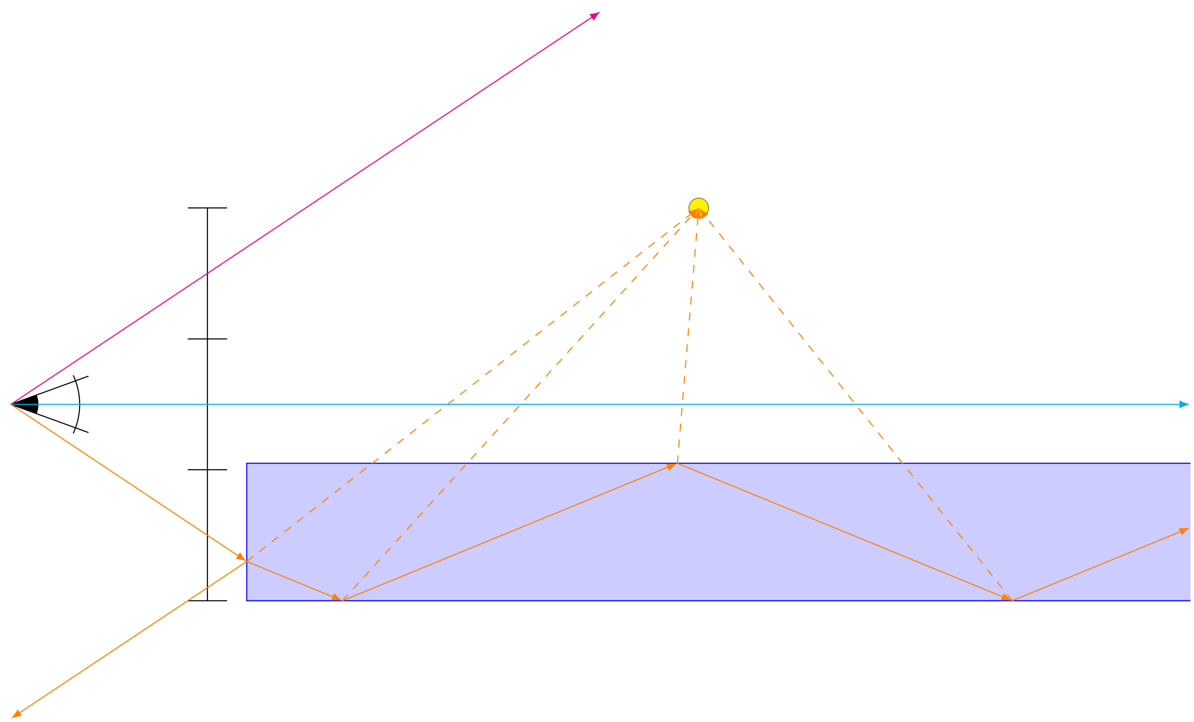
Problem 12 (4 points)

In each of the four examples below, control points are shown for a cubic Bezier curve. Sketch out approximately what these curves will look like.

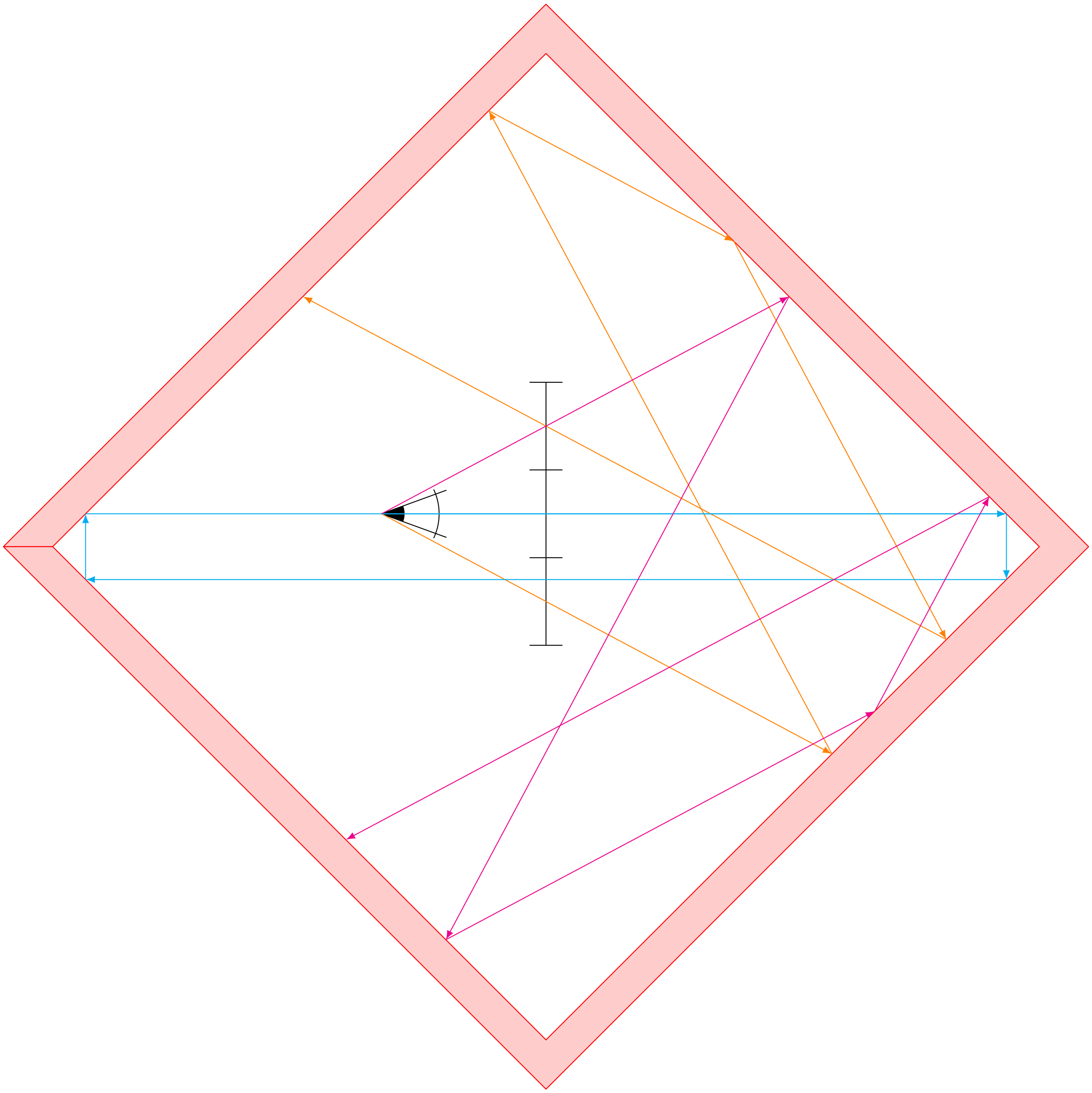


In the raytracing problems below, green objects are wood, red objects are reflective, and blue objects are transparent. The scenes are in 2D with a 1D image. yellow circles are point lights; the ray tracer supports shadows. Draw all of the rays that would be cast while raytracing each scene. Use a maximum recursion depth of 5. (Don't worry about precisely what counts as depth 5; I just care that recursion is being performed correctly when necessary and that important rays are not missing. There are no more than 20 rays in the "exact" solution.)

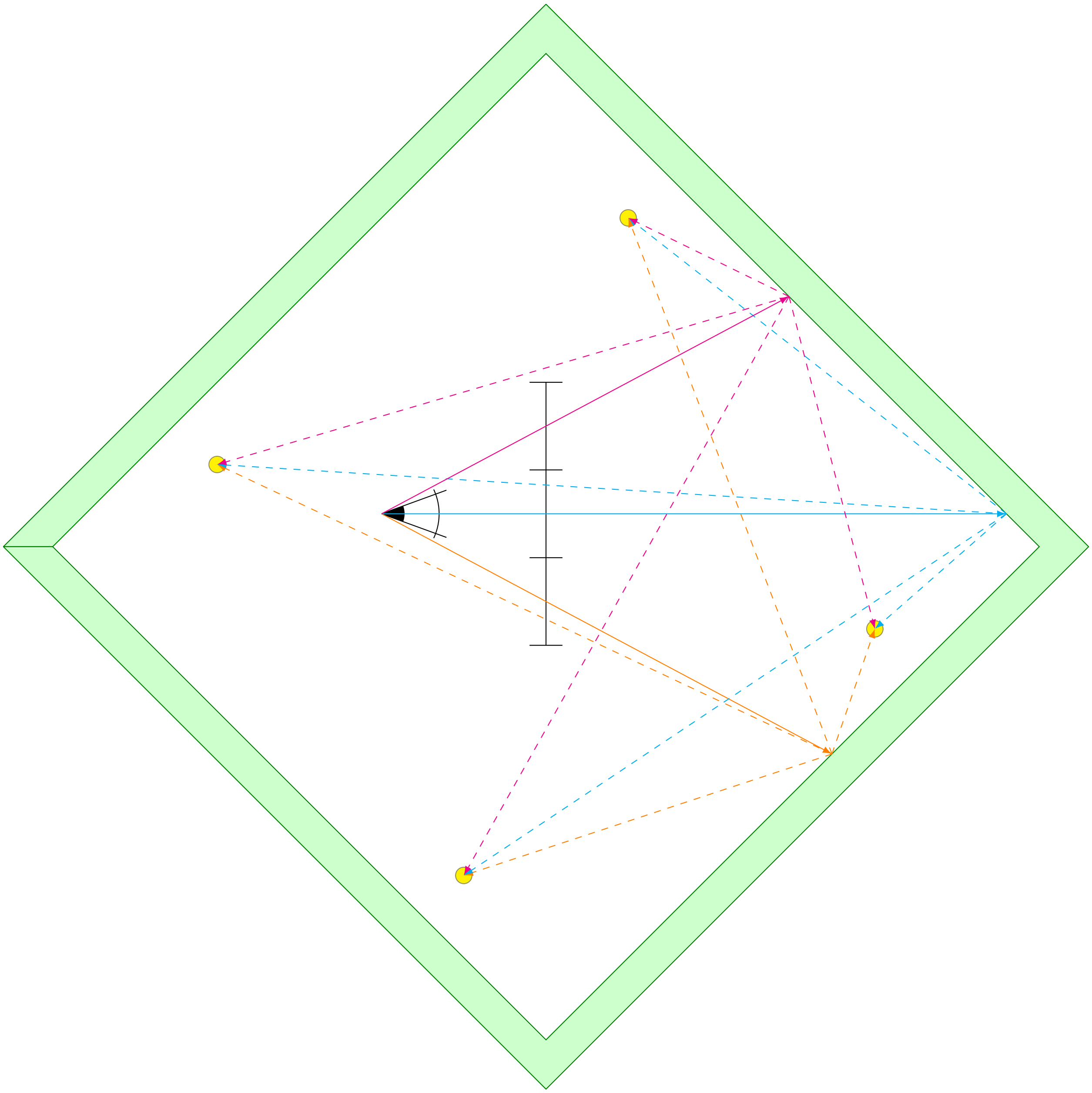
Problem 13 (5 points)



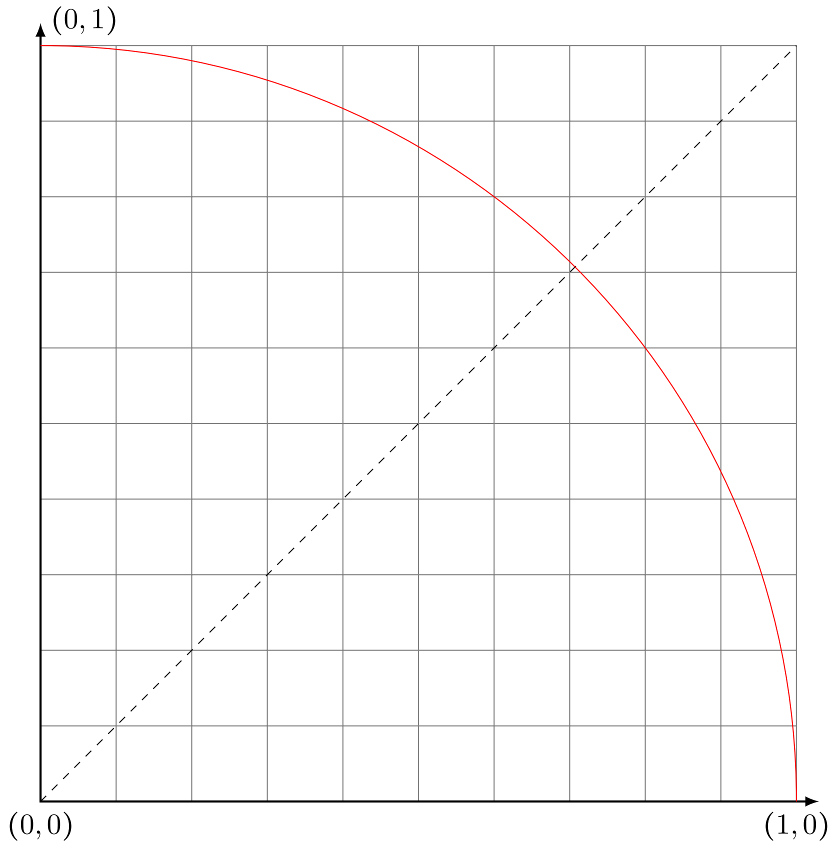
Problem 14 (5 points)



Problem 15 (5 points)



In the problems that follow, we will be approximating a quarter of the unit circle (shown below) using a cubic Bézier curve. Let $A = (a_0, a_1)$, $B = (b_0, b_1)$, $C = (c_0, c_1)$, and $D = (d_0, d_1)$ be the control points for the Bézier curve $P(t)$. In the problems that follow, we will calculate the eight degrees of freedom $a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1$. In many contexts (such as computer fonts), Bézier curves are used to approximate circles. This works well because the approximation is extremely good; the circle and its Bézier approximation in the figure would differ by less than the width of a human hair, far less than the width of the red line used to draw it.



Problem 16 (2 points)

What is $P(t)$ explicitly in terms of A, B, C, and D?

$P(t) = (1-t)^3 A + 3t(1-t)^2 B + 3t^2(1-t) C + t^3 D.$

Problem 17 (2 points)

Let's begin by forcing the curve to agree with the quarter-circle at the endpoints by requiring $P(0) = (1, 0)$ and $P(1) = (0, 1)$. Use this to eliminate as many of the components of the control points as possible. This should eliminate four of the eight components. Hint: this step should not be tedious.

$P(0) = A$, so $a_0 = 1$ and $a_1 = 0$. $P(1) = D$, so $d_0 = 0$ and $d_1 = 1$.

Problem 18 (2 points)

Next, lets force the curve to have the same slope as the quarter-circle at the endpoints. Use this to eliminate as many of the components of the control points as possible. This should eliminate two more of the components. Hint: this step should not be tedious.

The slope at $P(0)$ is along the vector from A to B. Since this should be vertical, $b_0 = a_0 = 1$. Similarly, $P(1)$ is along the vector from D to C. Since this should be horizontal, $c_1 = d_1 = 1$.

Problem 19 (2 points)

Next, we will require the curve to be symmetric about $x = y$ (shown dashed in the figure below). In particular, if $P(t) = (x, y)$ then $P(1-t) = (y, x)$. This should eliminate one more degree of freedom. Hint: this step should not be tedious.

The transformation $t \to 1 - t$ is equivalent to reversing the order of the control points. A and D already have the required symmetry. To make B and C have the symmetry, we need $c_0 = b_1$.

Problem 20 (3 points)

We are now left with just one degree of freedom, so we will need one additional constraint to solve for it. There are many ways to choose it. *(For example, we could make the area under the curve be* $\frac{\pi}{4}$*. We could minimize the maximum deviation of the curve from the circle. Forcing* $P(\frac{1}{3})$ *to lie of the unit circle also produces a very good approximation.)* In this problem, we will do something that is not quite as good but is much simpler. We will force $P(\frac{1}{2})$ to lie on the unit circle. That is, $\|P(\frac{1}{2})\|^2 = 1$. Use this to solve for the final degree of freedom. Hint: this step is straightforward but slightly tedious. You should get a quadratic equation in one variable. You will need to do the previous problems in order to solve this part.

Let $(x, y) = P(\frac{1}{2})$. From the symmetry imposed in the previous problem, we know that the $x = y$.

Thus, we only need to worry about the x component. $\|P(\frac{1}{2})\|^2 = 1$ reduces to $2x^2 = 1$.

$$
\begin{aligned}
(x, y) = P(t) &= (1-t)^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3t(1-t)^2 \begin{pmatrix} 1 \\ b \end{pmatrix} + 3t^2(1-t) \begin{pmatrix} b \\ 1 \end{pmatrix} + t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
x &= \frac{1}{8} + \frac{3}{8} + \frac{3}{8}b = \frac{(4+3b)}{8} \\
2x^2 &= 1 \\
2\left(\frac{(4+3b)}{8} \right)^2 &= 1 \\
\frac{(4+3b)}{8} &= \pm \frac{1}{\sqrt{2}} \\
4 + 3b &= \pm 4\sqrt{2} \\
b &= \frac{-4 \pm 4\sqrt{2}}{3} \\
b &= \frac{-4 + 4\sqrt{2}}{3} \qquad \text{Note: need } b > 0
\end{aligned}
$$

[1]

[1] Total points: 60