CS 230, Quiz 4

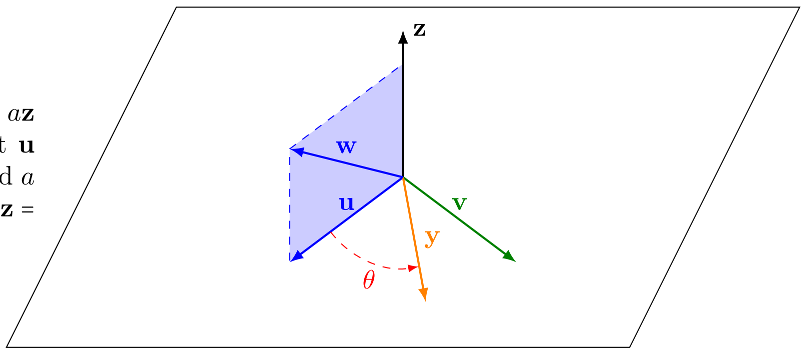
Solutions

You will have 20 minutes to complete this quiz. No books, notes, or other aids are permitted.

In the following sequence of problems, we will work out formulas for 3D rotations about a given axis $\mathbf{z}$ by a given angle θ. You may assume $\|\mathbf{z}\| = 1$. We will call the rotation matrix $\mathbf{R}$; our goal is to derive a formula for $\mathbf{R}$. We will do this by considering its effects on an arbitrary vector $\mathbf{w}$. That is, we will work out a formula for $\mathbf{Rw}$ that depends only on $\mathbf{w}$, $\mathbf{z}$, and θ. *All of the parts below may be completed independently and in any order. You may assume the results of earlier problems when completing later ones. In particular, some parts ask you to compute new variables; you may use those variables in later parts, even if you have not computed them. It is important to read earlier problems, since they contain information you may need to complete later ones.*

Problem 1 (1 points)

We begin by decomposing $\mathbf{w}$ into a part $a\mathbf{z}$ along the axis of rotation $\mathbf{z}$ and a part $\mathbf{u}$ orthogonal to $\mathbf{z}$, so that $\mathbf{w} = \mathbf{u} + a\mathbf{z}$. Find a in terms of $\mathbf{w}, \mathbf{z}, \theta$. You may assume $\mathbf{u} \cdot \mathbf{z} = 0$.



$\mathbf{w} = \mathbf{u} + a\mathbf{z}$. Using orthogonality and the fact that $\mathbf{z}$ is normalized, $\mathbf{w} \cdot \mathbf{z} = \mathbf{u} \cdot \mathbf{z} + a\mathbf{z} \cdot \mathbf{z} = a$.

Problem 2 (1 points)

Find $\mathbf{u}$ in terms of $\mathbf{w}, \mathbf{z}, \theta, a$.

$\mathbf{u} = \mathbf{w} - a\mathbf{z}$.

Problem 3 (1 points)

The plane shown has normal direction $\mathbf{z}$. Find another vector $\mathbf{v}$ in this plane, such that $\mathbf{u}\cdot\mathbf{v} = 0$ and $\mathbf{z}\cdot\mathbf{v} = 0$. Select your vector so that $\|\mathbf{u}\| = \|\mathbf{v}\|$. Your vector may depend on $\mathbf{w}, \mathbf{z}, \mathbf{u}, \theta, a$.

$\mathbf{v} = \mathbf{z} \times \mathbf{u}$.

Problem 4 (1 points)

Vectors in this plane will rotate by the full angle θ. Let $\mathbf{y} = \mathbf{R}\mathbf{u}$ be the vector that will be obtained by rotating $\mathbf{u}$ around the axis $\mathbf{z}$ by angle θ. Since this vector is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, we can write $\mathbf{y} = b\mathbf{u} + c\mathbf{v}$. Since rotations do not change lengths, we also know $\|\mathbf{y}\| = \|\mathbf{u}\|$. Find b in terms of θ. (Partial credit if you express in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \theta, a$, but if you have done this correctly, only θ will remain after simplification.)

$$\begin{aligned}
\mathbf{y} &= b\mathbf{u} + c\mathbf{v} \\
\mathbf{y}\cdot\mathbf{u} &= b\mathbf{u}\cdot\mathbf{u} + c\mathbf{v}\cdot\mathbf{u} \\
\|\mathbf{y}\|\|\mathbf{u}\|\cos\theta &= b\|\mathbf{u}\|^2 \\
b &= \cos\theta
\end{aligned}$$

Problem 5 (1 points)

Find c. There are two solutions; either is fine. Full credit if c is expressed in terms of b, θ only; partial credit if in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \theta, a, b$.

$$\begin{aligned}
\mathbf{y} &= b\mathbf{u} + c\mathbf{v} \\
\mathbf{y}\cdot\mathbf{y} &= (b\mathbf{u} + c\mathbf{v})\cdot(b\mathbf{u} + c\mathbf{v}) \\
\|\mathbf{y}\|^2 &= b^2\mathbf{u}\cdot\mathbf{u} + 2bc\mathbf{u}\cdot\mathbf{v} + c^2\mathbf{v}\cdot\mathbf{v} \\
\|\mathbf{y}\|^2 &= b^2\|\mathbf{u}\|^2 + c^2\|\mathbf{v}\|^2 \\
c^2 &= 1 - b^2 \\
c &= \pm\sqrt{1 - b^2} = \pm\sin\theta
\end{aligned}$$

Problem 6 (1 points)

Let $\mathbf{q} = \mathbf{R}\mathbf{z}$. Find $\mathbf{q}$ in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{y}, \theta, a, b, c$.

Rotations do not change vectors along the axis of rotation, so $\mathbf{q} = \mathbf{z}$.

Problem 7 (1 points)

Find $\mathbf{R}\mathbf{w}$ in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{q}, \theta, a, b, c$.

$\mathbf{R}\mathbf{w} = \mathbf{R}\mathbf{u} + a\mathbf{R}\mathbf{z} = \mathbf{y} + a\mathbf{q}$

Problem 8 (1 points)

Find a matrix $\mathbf{M}$ such that $\mathbf{M}\mathbf{s} = (\mathbf{s} \cdot \mathbf{z})\mathbf{z}$ for any vector $\mathbf{s}$. Your matrix $\mathbf{M}$ may only depend on the vector $\mathbf{z}$.

$\mathbf{M} = \mathbf{z}\mathbf{z}^T$.

Problem 9 (1 points)

Find a matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{s} = \mathbf{z} \times \mathbf{s}$ for any vector $\mathbf{s}$. Your matrix $\mathbf{A}$ may only depend on the vector $\mathbf{z}$. (You will need to write out $\mathbf{A}$ in terms of the components of $\mathbf{z}$.)

$\mathbf{M} = \mathbf{z}\mathbf{z}^T$.