CS 230, Quiz 6

Solutions

You will have 20 minutes to complete this quiz. No books, notes, or other aids are permitted.

You are given these formulas from class:

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\begin{array}{lllll}
\mathbf{x}_i = \mathbf{x} + \mathbf{r}_i & \mathbf{v}_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i & \mathbf{r}_i = \mathbf{R}\hat{\mathbf{r}}_i & \mathbf{u}^*\mathbf{w} = \mathbf{u} \times \mathbf{w} & \mathbf{u}^{*T}\mathbf{w} = \mathbf{w} \times \mathbf{u} \\
m = \sum_i m_i & \mathbf{I} = \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^{*T} & \hat{\mathbf{I}} = \sum_i m_i \hat{\mathbf{r}}_i^* \hat{\mathbf{r}}_i^{*T} & \mathbf{p} = \sum_i m_i \mathbf{v}_i & \mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \\
\dot{\mathbf{R}} = \boldsymbol{\omega}^* \mathbf{R} & \dot{\mathbf{x}} = \mathbf{v} & \dot{\mathbf{p}} = \mathbf{f} & \dot{\mathbf{L}} = \boldsymbol{\tau} & \boldsymbol{\delta} = \text{identity matrix} \\
\mathbf{0} = \sum_i m_i \mathbf{r}_i & \mathbf{0} = \sum_i m_i \hat{\mathbf{r}}_i & \mathbf{L} = \mathbf{I}\boldsymbol{\omega} & \mathbf{p} = m\mathbf{v} & \mathbf{u}_i^* \mathbf{u}_i^{*T} = \boldsymbol{\delta}(\mathbf{u}^T\mathbf{u}) - \mathbf{u}\mathbf{u}^T
\end{array}
$$

Problem 1 (1 points)

The total kinetic energy of a rigid body is just the total energy of its parts,

$$E = \sum_i \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i.$$

Derive a formula for E that depends only on the state of the rigid body ($\mathbf{x}$, $\mathbf{R}$, $\mathbf{v}$, $\boldsymbol{\omega}$, m, $\mathbf{I}$, $\mathbf{p}$, $\mathbf{L}$).

$$
\begin{aligned}
E &= \sum_i \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i \\
&= \sum_i \frac{1}{2} m_i (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i) \\
&= \sum_i \frac{1}{2} m_i \mathbf{v} \cdot \mathbf{v} + \sum_i m_i \mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}_i) + \sum_i \frac{1}{2} m_i (\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}_i) \\
&= \frac{1}{2}\left(\sum_i m_i\right) \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i\right) + \frac{1}{2} \sum_i m_i (\mathbf{r}_i^{*T} \boldsymbol{\omega})^T (\mathbf{r}_i^{*T} \boldsymbol{\omega}) \\
&= \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{0}) + \frac{1}{2} \boldsymbol{\omega}^T \left(\sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^{*T}\right) \boldsymbol{\omega} \\
&= \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega}
\end{aligned}
$$