Eigenproblem of 2×2 Symmetrix Matrix

June 30, 2007

Consider the symmetric matrix

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$$

Let $a = (x_{11} + x_{22})/2$, $b = (x_{11} - x_{22})/2$, $c = x_{21}$, $m = \sqrt{b^2 + c^2}$ then we have

$$\begin{bmatrix} a+b & c \\ c & a-b \end{bmatrix}$$

In the case m is close to zero, or rather in the case where c and b are roughly zero, then we return a as both the eignevalues and $(1, 0)$ and $(0, 1)$ as the eigenvectors. We have $k = a^2 - b^2 - c^2 = \det A$. Then we have the following cases for the eigenvalues

1. If $a \geq 0$ then $\lambda_1 = a + m$ and $\lambda_2 = \frac{k}{a+m}$
2. Else $a < 0$ then $\lambda_1 = \frac{k}{a-m}$ and $\lambda_2 = a - m$

For eigenvectors we have these cases:

1. If $b \geq 0$ then $v_1 = \begin{bmatrix} m+b \\ c \end{bmatrix}$ and $v_2 = \begin{bmatrix} -c \\ m+b \end{bmatrix}$
2. Else If $b < 0$ then $v_1 = \begin{bmatrix} -c \\ b-m \end{bmatrix}$ and $v_2 = \begin{bmatrix} b-m \\ c \end{bmatrix}$

Note that the eigenvalues are ordered in terms of value as $\lambda_1 = a + m$ and $\lambda_2 = a - m$ where $m \geq 0$.

Here we describe the computation hazards and avoided hazards:

- Computing a when $x_{11} \approx -x_{22}$
- Computing b when $x_{11} \approx x_{22}$
- Computing m is safe because you are adding two positive values
- The same holds in the eigenvalue computations for the additions in the eigenvalue if $a \geq 0$ and for the subtraction if $a < 0$

- The same holds for the eigenvector additions of $m + b$ when $b \geq 0$ and subtraction $b - m$ when $b < 0$.
- The eigenvalue divisions are fine as the eigenvalues have magnitudes that are at least as large as m.
- Computing k when the matrix is roughly singular is an issue.
- The normalization is safe because m is bounded greater than zero and in this case $b + m \geq m > 0$ and in the other case $b - m \leq -m < 0$ thus we are bounded away from the zero vector.