Rigid Grouping

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Suppose we have a system of particles with masses, positions, and velocities $m_i, \mathbf{x}_i, \mathbf{v}_i$. This system has center of mass $\mathbf{x} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i}$, and linear and angular momenta:

- Linear momentum: $\mathbf{p} = \sum m_i \mathbf{v}_i$
- Angular momentum about $\mathbf{x}$: $\mathbf{L} = \sum \mathbf{r}_i \times m_i \mathbf{v}_i$

Let $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}$. Note that $\sum m_i \mathbf{r}_i = 0$ (easy to show using definition of $\mathbf{x}$). If these particles were moving as a rigid group with linear velocity $\mathbf{v}$ and angular velocity ω then $\mathbf{v}_i = \mathbf{v} + \omega \times \mathbf{r}_i$ and we would have

- Linear momentum: $\mathbf{p}' = \sum m_i (\mathbf{v} + \omega \times \mathbf{r}_i) = \sum m_i \mathbf{v} + \underbrace{\omega \times \sum m_i \mathbf{r}_i}_{=0\, since\, \sum m_i \mathbf{r}_i = 0} = \sum m_i \mathbf{v}$
- Angular momentum: $\mathbf{L}' = \sum \mathbf{r}_i \times m_i (\mathbf{v} + \omega \times \mathbf{r}_i) = \underbrace{\sum m_i \mathbf{r}_i \times \mathbf{v}}_{=0\, since\, \sum m_i \mathbf{r}_i = 0} + \sum m_i \mathbf{r}_i \times (\omega \times \mathbf{r}_i)$
  - Note $\mathbf{r}_i \times (\omega \times \mathbf{r}_i) = (\mathbf{r}_i \cdot \mathbf{r}_i)\, \omega - (\mathbf{r}_i \cdot \omega)\, \mathbf{r}_i = \left(\mathbf{r}_i^T \mathbf{r}_i - \mathbf{r}_i \mathbf{r}_i^T\right) \omega$
  - Letting $I = \sum m_i \left(\mathbf{r}_i^T \mathbf{r}_i - \mathbf{r}_i \mathbf{r}_i^T\right)$, we get $\mathbf{L}' = I\omega$

We want the rigid group to conserve momentum, hence we can solve for $\mathbf{v}$ and ω:

$\mathbf{p} = \mathbf{p}' \Rightarrow \sum m_i \mathbf{v}_i = \sum m_i \mathbf{v} \Rightarrow \mathbf{v} = \frac{\sum m_i \mathbf{v}_i}{\sum m_i}$

$\mathbf{L} = \mathbf{L}' \Rightarrow \sum \mathbf{r}_i \times m_i \mathbf{v}_i = I\omega \Rightarrow \omega = I^{-1} \left(\sum \mathbf{r}_i \times m_i \mathbf{v}_i\right)$