1 Finite Elements

Symbol	Definition	Dimensions	Units	Meaning
$\mathbf{1}$	$\mathbf{1} = \begin{pmatrix}1 & \cdots & 1\end{pmatrix}^T$	d	1	All ones vector
$\mathbf{S}$	$\mathbf{S} = \begin{pmatrix}\mathbf{I} & -\mathbf{1}\end{pmatrix}$	$d \times (d+1)$	1	Scatter matrix
$\mathbf{X}_m$	$\begin{pmatrix}X_1 & X_2 & X_3 & X_4\end{pmatrix}_m$	$d \times (d+1)$	m	Material space element node positions
$\mathbf{X}$	$\begin{pmatrix}X_1 & X_2 & X_3 & X_4\end{pmatrix}$	$d \times (d+1)$	m	World space element node positions
$\mathbf{D}_m$	$\mathbf{D}_m = \mathbf{X}_m \mathbf{S}^T$	$d \times d$	m	Material space relative node positions
$\mathbf{D}_s$	$\mathbf{D}_s = \mathbf{X}\mathbf{S}^T$	$d \times d$	m	World space relative node positions
$\mathbf{F}$	$\mathbf{F} = \mathbf{D}_s \mathbf{D}_m^{-1}$	$d \times d$	1	Deformation gradient
J	$J = \det(\mathbf{F})$	scalar	1	Jacobian
I_1	$I_1 = \operatorname{tr}(\mathbf{F}\mathbf{F}^T)$	scalar	1	First invariant
V	$V = \frac{1}{d!}\det(\mathbf{D}_m)$	scalar	m^d	Material space element volume
$\mathbf{N}$	$\mathbf{N} = V\mathbf{D}_m^{-T}$	$d \times d$	m^{d-1}	Material space area weighted normals
ψ	-	scalar	$kg\, m^{2-d} s^{-1}$	Energy density
ϕ	$\phi = V\psi$	scalar	$kg\, m^2 s^{-1}$	Potential energy of element
$\mathbf{f}$	$\mathbf{f} = -\frac{\partial \phi}{\partial \mathbf{X}}$	$d \times (d+1)$	$kg\, ms^{-1}$	Force on element nodes
$\mathbf{P}$	$\mathbf{f} = -\mathbf{P}\mathbf{N}\mathbf{S}^T$	$d \times d$	$kg\, m^{2-d} s^{-1}$	First Piola-Kirchoff stress

Consider that a small position change $\delta\mathbf{X}$ is made. Then

$$
\begin{aligned}
0 &= \mathbf{f} : \delta\mathbf{X} + \delta\phi \\
&= \operatorname{tr}(\mathbf{f}\delta\mathbf{X}^T) + \delta\phi \\
&= \operatorname{tr}(-\mathbf{P}\mathbf{N}\mathbf{S}^T\delta\mathbf{X}^T) + V\delta\psi \\
&= -V\operatorname{tr}(\mathbf{P}\mathbf{D}_m^{-T}\mathbf{S}^T\delta\mathbf{X}^T) + V\delta\psi \\
\delta\psi &= \operatorname{tr}(\mathbf{P}\mathbf{D}_m^{-T}\mathbf{S}^T\delta\mathbf{X}^T) \\
&= \operatorname{tr}(\mathbf{P}\delta(\mathbf{D}_m^{-T}\mathbf{S}^T\mathbf{X}^T)) \\
&= \operatorname{tr}(\mathbf{P}\delta(\mathbf{D}_m^{-T}\mathbf{D}_s^T)) \\
&= \operatorname{tr}(\mathbf{P}\delta\mathbf{F}^T) \\
&= \mathbf{P} : \delta\mathbf{F} \\
\mathbf{P} &= \frac{\partial\psi}{\partial\mathbf{F}}
\end{aligned}
$$

2 Potential Energy for Neo Hookean

Let $\psi = \frac{\mu}{2}(I_1 - d) - \mu \ln J + \frac{\lambda}{2}\ln^2 J$.

$$
\begin{aligned}
\frac{\partial J}{\partial \mathbf{F}} &= \frac{\partial}{\partial \mathbf{F}}\det(\mathbf{F}) = \det(\mathbf{F})\mathbf{F}^{-1} = J\mathbf{F}^{-1} \\
\frac{\partial I_1}{\partial \mathbf{F}} &= \frac{\partial}{\partial \mathbf{F}}\operatorname{tr}(\mathbf{F}\mathbf{F}^T) = 2\mathbf{F} \\
\frac{\partial}{\partial \mathbf{F}} &= \frac{\partial}{\partial \mathbf{F}}\ln J = \mathbf{F}^{-1} \\
\mathbf{P} = \frac{\partial\psi}{\partial\mathbf{X}} &= \frac{\partial}{\partial\mathbf{X}}\left(\frac{\mu}{2}(I_1 - d) - \mu\ln J + \frac{\lambda}{2}\ln^2 J\right) \\
&= \frac{\mu}{2}\frac{\partial I_1}{\partial\mathbf{X}} - \mu\frac{\partial}{\partial\mathbf{F}}\ln J + \lambda\ln J\frac{\partial}{\partial\mathbf{F}}\ln J \\
&= \mu(\mathbf{F} - \mathbf{F}^{-1}) + \lambda\ln J\mathbf{F}^{-1}
\end{aligned}
$$