3.5 Canonical Forms

In general, the unique algebraic expression for any Boolean function can be obtained from its truth table by using an OR operator to combined all minterms for which the function is equal to 1.

A **minterm**, denoted as m_i, where $0 \le i < 2^n$, is a product (AND) of the n variables in which each variable is complemented if the value assigned to it is 0, and uncomplemented if it is 1.

1-minterms = minterms for which the function $F = 1$.

0-minterms = minterms for which the function $F = 0$.

Any Boolean function can be expressed as a sum (OR) of its **1-minterms**. A shorthand notation:

F(list of variables) = Σ(list of 1-minterm indices)

Ex. $F = x'yz + xy'z + xyz' + xyz$

$= m_3 + m_5 + m_6 + m_7$

or

$F(x, y, z) = \Sigma(3, 5, 6, 7)$

The complement of the function can be expressed as a sum (OR) of its **0-minterms**. A shorthand notation:

F(list of variables) = Σ(list of 0-minterm indices)

Ex. $F' = x'y'z' + x'y'z + x'yz' + xy'z'$

$= m_0 + m_1 + m_2 + m_4$

or

$F'(x, y, z) = \Sigma(0, 1, 2, 4)$

Ex. Express the Boolean function $F = x + yz$ as a sum of minterms.

Solution: This function has three variables: x, y, and z. All terms must have these three variables. Thus, we need to expand the first term by ANDing it with $(y + y')(z + z')$, and we expand the second term with $(x + x')$ to get

$$\begin{aligned} F &= x + yz \\ &= x(y + y')(z + z') + (x + x')yz \\ &= xyz + xyz' + xy'z + xy'z' + xyz + x'yz \\ &= x'yz + xy'z' + xy'z + xyz' + xyz \\ &= m_3 + m_4 + m_5 + m_6 + m_7 \\ &= \Sigma(3, 4, 5, 6, 7) \end{aligned}$$

Ex. Convert the Boolean function $F = x + yz$ into a sum of minterms by using a truth table.

Solution: First, we derive the truth table 3.10, then from the table we get $F = 1$ for minterms m_3, m_4, m_5, m_6, and m_7. Therefore, $F = m_3 + m_4 + m_5 + m_6 + m_7$, which is the same as above when we used term expansion.

x	y	z	Minterms	Notation
0	0	0	$x'y'z'$	m_0
0	0	1	$x'y'z$	m_1
0	1	0	$x'yz'$	m_2
0	1	1	$x'yz$	m_3
1	0	0	$xy'z'$	m_4
1	0	1	$xy'z$	m_5
1	1	0	xyz'	m_6
1	1	1	xyz	m_7

Table 3.9

$F = x'yz + xy'z + xyz' + xyz$

$F' = x'y'z' + x'y'z + x'yz' + xy'z'$

x	y	z	Minterms	F	F'
0	0	0	$m_0=x'y'z'$	0	1
0	0	1	$m_1=x'y'z$	0	1
0	1	0	$m_2=x'yz'$	0	1
0	1	1	$m_3=x'yz$	1	0
1	0	0	$m_4=xy'z'$	0	1
1	0	1	$m_5=xy'z$	1	0
1	1	0	$m_6=xyz'$	1	0
1	1	1	$m_7=xyz$	1	0

Table 3.8

$F = x + yz$

x	y	z	Minterms	F	F'
0	0	0	$m_0=x'y'z'$	0	1
0	0	1	$m_1=x'y'z$	0	1
0	1	0	$m_2=x'yz'$	0	1
0	1	1	$m_3=x'yz$	1	0
1	0	0	$m_4=xy'z'$	1	0
1	0	1	$m_5=xy'z$	1	0
1	1	0	$m_6=xyz'$	1	0
1	1	1	$m_7=xyz$	1	0

Table 3.10

A **maxterm**, denoted as M_i, where $0 \le i < 2^n$, is a sum (OR) of the n variables (literals) in which each variable is complemented if the value assigned to it is 1, and uncomplemented if it is 0.

1-maxterms = maxterms for which the function $F = 1$.

0-maxterms = maxterms for which the function $F = 0$.

x	y	z	Maxterms	Notation
0	0	0	$x + y + z$	M_0
0	0	1	$x + y + z'$	M_1
0	1	0	$x + y' + z$	M_2
0	1	1	$x + y' + z'$	M_3
1	0	0	$x' + y + z$	M_4
1	0	1	$x' + y + z'$	M_5
1	1	0	$x' + y' + z$	M_6
1	1	1	$x' + y' + z'$	M_7

Table 3.11

Any Boolean function can be expressed as a product (AND) of its **0-maxterms**. A shorthand notation:

F(list of variables) = Π(list of 0-maxterm indices)

Ex. $F = x+y+z \bullet x+y+z' \bullet x+y'+z \bullet x'+y+z$

$= M_0 \bullet M_1 \bullet M_2 \bullet M_4$

or

$F(x, y, z) = \Pi(0, 1, 2, 4)$

The complement of the function can be expressed as a product (AND) of its **1-maxterms**. A shorthand notation:

F(list of variables) = Π(list of 1-maxterm indices)

Ex. $F' = x+y'+z' \bullet x'+y+z' \bullet x'+y'+z \bullet x'+y'+z'$

$= M_3 \bullet M_5 \bullet M_6 \bullet M_7$

or

$F'(x, y, z) = \Pi(3, 5, 6, 7)$

$F = x+y+z \bullet x+y+z' \bullet x+y'+z \bullet x'+y+z$

$F' = x+y'+z' \bullet x'+y+z' \bullet x'+y'+z \bullet x'+y'+z'$

x	y	z	Maxterms	F	F'
0	0	0	$M_0 = x + y + z$	0	1
0	0	1	$M_1 = x + y + z'$	0	1
0	1	0	$M_2 = x + y' + z$	0	1
0	1	1	$M_3 = x + y' + z'$	1	0
1	0	0	$M_4 = x' + y + z$	0	1
1	0	1	$M_5 = x' + y + z'$	1	0
1	1	0	$M_6 = x' + y' + z$	1	0
1	1	1	$M_7 = x' + y' + z'$	1	0

Definition: Any Boolean function that is expressed as a sum of minterms or as a product of maxterms is said to be in its **canonical form**.

To **convert** from one canonical form to another, interchange the symbols Σ and Π, and list the index numbers that were excluded from the original form.

To convert from one canonical form to its **dual**, interchange the symbols Σ and Π, and list the index numbers from the original form.

Ex. $F = m_3 + m_5 + m_6 + m_7 = \Sigma(3, 5, 6, 7)$

$= x'\,y\,z + x\,y'\,z + x\,y\,z' + x\,y\,z$

equivalent

$= M_0 \bullet M_1 \bullet M_2 \bullet M_4 = \Pi(0, 1, 2, 4)$

$= x+y+z \bullet x+y+z' \bullet x+y'+z \bullet x'+y+z$

duals

$F' = m_1 + m_1 + m_2 + m_4 = \Sigma(0, 1, 2, 4)$

$= x'\,y'\,z' + x'\,y'\,z + x'\,y\,z' + x\,y'\,z'$

equivalent

$= M_3 \bullet M_5 \bullet M_6 \bullet M_7 = \Pi(3, 5, 6, 7)$

$= x+y'+z' \bullet x'+y+z' \bullet x'+y'+z \bullet x'+y'+z'$

To convert an expression to its canonical form, all terms must contain all variables.

- To get the sum of minterms, we expand each term by ANDing it with (v + v') for every missing variable v in that term.
- To get the product of maxterms, we expand each term by ORing it with (v v') for every missing variable v in that term.

Ex. Express the Boolean function $F = x + y\,z$ as a product of maxterms.

Solution: First, we need to convert the function into the product-of-OR terms by using the distributive law as follows:

$F = x + y\,z$
$= (x + y)\,(x + z)$ distributive law
$= (x + y + z\,z')\,(x + y\,y' + z)$ expand 1st term by ORing it with $z\,z'$, and 2nd term with $y\,y'$
$= (x + y + z)\,(x + y + z')\,\cancel{(x + y + z)}\,(x + y' + z)$
$= M_0 \bullet M_1 \bullet M_2$
$= \Pi(0, 1, 2)$

Ex. Express $F\,' = (x + y\,z)'$ as a product of maxterms.

Solution:

$F\,' = (x + y\,z)'$
$= x' \bullet (y' + z')$ dual
$= (x' + y\,y' + z\,z')\,(x\,x' + y' + z')$ expand 1st term by ORing it with $y\,y'$ and $z\,z'$, and 2nd term with $x\,x'$
$= (x' + y + z)\,(x' + y + z')\,(x' + y' + z)\,(x' + y' + z')\,(x + y' + z')\,\cancel{(x' + y' + z')}$
$= M_4 \bullet M_6 \bullet M_5 \bullet M_7 \bullet M_3$
$= \Pi(3, 4, 5, 6, 7)$

Ex. Express $F\,' = (x + y\,z)'$ as a sum of minterms.

Solution:

$F\,' = (x + y\,z)'$
$= x' \bullet (y' + z')$ dual
$= (x'\,y') + (x'\,z')$ distributive law
$= x'\,y'\,(z + z') + x'\,(y + y')\,z'$ expand 1st term by ANDing it with $(z + z')$, and 2nd term with $(y + y')$
$= x'\,y'\,z + x'\,y'\,z' + x'\,y\,z' + \cancel{x'\,y'\,z'}$
$= m_1 + m_0 + m_2$
$= \Sigma(0, 1, 2)$

Example: Given the function as defined in the truth table, express F using sum of minterms and product of maxterms, and express F' using sum of minterms and product of maxterms.

x	y	z	Minterms	Maxterms	F	F'
0	0	0	$m_0 = x' y' z'$	$M_0 = x + y + z$	0	1
0	0	1	$m_1 = x' y' z$	$M_1 = x + y + z'$	1	0
0	1	0	$m_2 = x' y z'$	$M_2 = x + y' + z$	0	1
0	1	1	$m_3 = x' y z$	$M_3 = x + y' + z'$	1	0
1	0	0	$m_4 = x y' z'$	$M_4 = x' + y + z$	1	0
1	0	1	$m_5 = x y' z$	$M_5 = x' + y + z'$	0	1
1	1	0	$m_6 = x y z'$	$M_6 = x' + y' + z$	0	1
1	1	1	$m_7 = x y z$	$M_7 = x' + y' + z'$	0	1

Solution:

F as sum of minterms:

$$\begin{aligned} F &= m_1 + m_3 + m_4 \\ &= \Sigma(1, 3, 4) \\ &= x' y' z + x' y z + x y' z' \end{aligned}$$

F as product of maxterms:

$$\begin{aligned} F &= M_0 \bullet M_2 \bullet M_5 \bullet M_6 \bullet M_7 \\ &= \Pi(0, 2, 5, 6, 7) \\ &= (x + y + z)(x + y' + z)(x' + y + z')(x' + y' + z)(x' + y' + z') \end{aligned}$$

duals

F' as sum of minterms:

$$\begin{aligned} F' &= m_0 + m_2 + m_5 + m_6 + m_7 \\ &= \Sigma(0, 2, 5, 6, 7) \\ &= x' y' z' + x' y z' + x y' z + x y z' + x y z \end{aligned}$$

F' as product of maxterms:

$$\begin{aligned} F' &= M_1 \bullet M_3 \bullet M_4 \\ &= \Pi(1, 3, 4) \\ &= (x + y + z')(x + y' + z')(x' + y + z) \end{aligned}$$

To transform $\Pi(1, 3, 4)$ to $\Sigma(0, 2, 5, 6, 7)$:

$$\begin{aligned} \Pi(1, 3, 4) &= (x + y + z')(x + y' + z')(x' + y + z) \\ &= \cancel{x\,x\,x'} + x\,x\,y + x\,x\,z + \\ &\quad\; \cancel{x\,y'\,x'} + \cancel{x\,y'\,y} + x\,y'\,z + \\ &\quad\; \cancel{x\,z'\,x'} + x\,z'\,y + \cancel{x\,z'\,z} + \\ &\quad \cancel{y\,x\,x'} + y\,x\,y + y\,x\,z + \\ &\quad\; \cancel{y\,y'\,x'} + \cancel{y\,y'\,y} + \cancel{y\,y'\,z} + \\ &\quad\; y\,z'\,x' + y\,z'\,y + \cancel{y\,z'\,z} + \\ &\quad \cancel{z'\,x\,x'} + z'\,x\,y + \cancel{z'\,x\,z} + \\ &\quad\; z'\,y'\,x' + \cancel{z'\,y'\,y} + \cancel{z'\,y'\,z} + \\ &\quad\; z'\,z'\,x' + z'\,z'\,y + \cancel{z'\,z'\,z} \\ &= x' y' z' + x' y z' + x y' z + x y z' + x y z \end{aligned}$$

get every possible combinations

can eliminate terms with vv' because it is 0