

Parallel and (Nearly) Work-Efficient Dynamic Programming

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ABSTRACT

The idea of dynamic programming (DP), proposed by Bellman in the 1950s, is one of the most important algorithmic techniques. However, in parallel, many fundamental and sequentially simple problems become more challenging, and open to a (nearly) work-efficient solution (i.e., the work is off by at most a polylogarithmic factor over the best sequential solution). In fact, sequential DP algorithms employ many advanced optimizations such as decision monotonicity or special data structures, and achieve better work than straightforward solutions. Many such optimizations are inherently sequential, which creates extra challenges for a parallel algorithm to achieve the same work bound.

The goal of this paper is to achieve (nearly) work-efficient parallel DP algorithms by parallelizing classic, highly-optimized and practical sequential algorithms. We show a general framework called the *Cordon Algorithm* for parallel DP algorithms, and use it to solve several classic problems. Our selection of problems includes Longest Increasing Subsequence (LIS), sparse Longest Common Subsequence (LCS), convex/concave generalized Least Weight Subsequence (LWS), Optimal Alphabetic Tree (OAT), and more. We show how the Cordon Algorithm can be used to achieve the same level of optimization as the sequential algorithms, and achieve good parallelism. Many of our algorithms are conceptually simple, and we show some experimental results as proofs-of-concept.

CCS CONCEPTS

• Theory of computation → Parallel algorithms.

KEYWORDS

Parallel Algorithms, Dynamic Programming, Longest Increasing Subsequence, Longest Common Subsequence, Least Weighted Sum, Optimal Alphabetic Tree, Decision Monotonicity

1 INTRODUCTION

The idea of dynamic programming (DP), since proposed by Richard Bellman in the 1950s [13], has been extensively used in algorithm design, and is one of the most important algorithmic techniques. It is covered in classic textbooks and basic algorithm classes, and is widely used in research and industry. The goal of this paper is to achieve (*nearly*) *work-efficient* (defined below) and *parallel* DP algorithms based on parallelizing classic, highly-optimized and practical sequential algorithms.

At a high level, a DP algorithm computes the *DP values* for a set of *states* (labeled by integers) by a *recurrence*. The recurrence specifies a set of *transitions* from state j to state i , i.e., how $D[j]$ can be used to compute $D[i]$ ¹. We call j a *decision* at i . $D[i]$ is then computed by taking the best (minimum or maximum) among

¹More generally, a transition may compute $D[i]$ from multiple other states. All algorithms in this paper only requires one state j in the transition to compute $D[i]$.

all decisions, which we call the *best decision* at i . Throughout the paper, we will use i^* to denote the best decision of state i . We introduce more concepts about DP in Sec. 2.

One can view the states and transitions as a directed acyclic graph (DAG), which we refer to as a *DP DAG*. In this DAG, each vertex is a state, and an edge j to i denotes a transition from j to i . Since such an edge indicates that computing $D[i]$ logically requires $D[j]$, we also call it a *dependency*, and say i *depends on* j . Sequentially, we can compute all states based on a topological ordering. For simplicity, we always assume that the (integer) order of the states is a valid topological ordering.

Unfortunately, many DP algorithms (even some simple ones sequentially) are hard to parallelize, and are especially hard to achieve work-efficiency (the work asymptotically matches the best sequential algorithm) or even near work-efficiency (off by a polylogarithmic factor). We note that on today’s multicore machines with tens to hundreds of processors, achieving low work is one of the most crucial objectives for designing *practical* parallel algorithms. One particularly intriguing and somewhat ironic challenge for achieving work-efficient parallel DP algorithms is that, sequential algorithms are extremely well-optimized. In many cases, an optimized DP algorithm does not need to process all edges (transitions) in the DP DAG; some even do not need to process all vertices (states). We review the literature at the end of this section. In fact, almost all textbook DP solutions can be optimized to achieve lower work than the straightforward solution that directly computes the DP values of all states based on the recurrence. Such examples include longest increasing subsequence (LIS), (sparse) longest common subsequence (LCS), (convex/concave) least weight subsequence (LWS), and many others discussed in this paper.

One typical DP optimization that is both theoretically elegant and practically useful is *decision monotonicity (DM)*. At a high level, DM indicates that two states i and $j > i$ must have their best decisions $j^* \geq i^*$, called the *convex case*², or the *concave case* where either $j^* \leq i^*$ or $j^* \geq i$ (see Fig. 1). Hence, when finding the best decision for state j , one can narrow down the possible range of j^* based on the known best decisions of previous states, and thus avoid processing all transitions. DM has been widely studied in the sequential setting (e.g., [35, 39, 41, 42, 60, 60, 61]), and is also closely related to concepts such as quadrangle inequalities [89, 90] and Monge property [76]. Sequentially, using DM saves a polynomial factor than the naïve DP algorithm in many applications [1, 2, 16, 40, 64, 75, 87]. In the parallel setting, however, among the papers we know of [21, 23, 29, 43, 52, 67, 78], most from the 90s, very few of them take advantage of DM to reduce work. Indeed, none of them are work-efficient, and most of them have a polynomial overhead, which limits their potential applicability on today’s multicore machines. The only nearly work-efficient results [21, 23] focus on the concave case of one specific problem.

²The definitions of convexity and concavity are interchanged in some other papers.

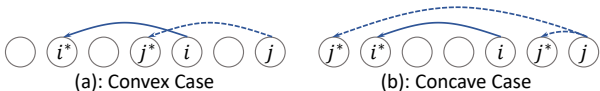


Figure 1: Convex and concave decision monotonicity. (a). Convexity: for two states $i < j$, their best decisions satisfy $i^* \leq j^*$. (b). Concavity: for two states $i < j$, their best decisions satisfy either $j^* \leq i^*$ or $j^* \geq i$.

The challenge of using DM in parallel lies in two aspects. First, sequentially we skip the transitions for state i by observing the best decisions of all states before i . When multiple states are processed in parallel, they cannot see each other’s best decision, making it hard to skip the same set of “useless” transitions as in the sequential setting. Second, many classic sequential DP algorithms with DM relies on efficient data structures such as monotonic queues, which are inherently sequential. Achieving the same work bound in parallel also requires careful redesign of the underlying data structures.

In this paper, we study parallel DP algorithms to achieve the same work as highly-optimized sequential algorithms. Given a sequential algorithm Γ with certain optimizations, our goal for work-efficiency is to process (asymptotically) the same number of transitions and states as in Γ . Regarding parallelism, we hope to achieve the best possible parallelism indicated by the transitions processed by Γ . We formalize our goals in Sec. 2.3. Our solution is based on an algorithmic framework that generally applies to almost *all* DP algorithms. We call this framework the **Cordon Algorithm**. At a high level, our framework specifies how to *correctly* identify a subset of states that do not depend on each other and process them in parallel. We then present how to do so *efficiently* for specific problems. To achieve work-efficiency, our key ideas are two-fold. First, many of our algorithms use *prefix-doubling* to bound the additional work on processing unnecessary states. Second, we design *new parallel data structures* to skip unnecessary transitions.

This paper studies general approaches for parallel DP, with a special focus on applying the non-trivial, effective optimizations found in the sequential context to parallel algorithms. We select classic DP problems and their optimized sequential solutions, and parallelize them using novel techniques. Our framework unifies one existing parallel LIS algorithm [45], and provides new parallel algorithms for various problems such as sparse LCS, convex/concave generalized LWS and GAP edit distance (GAP), and optimal alphabetic tree (OAT). All the algorithms are (nearly) work-efficient with non-trivial parallelism. Among them, we highlight our contributions on parallelizing the DP algorithms with decision monotonicity. The core of our idea is a parallel algorithm for convex/concave generalized LWS. We apply it to other problems such as GAP and OAT, and achieve new theoretical results. For OAT, we partially solve the open problem in [70] by providing a work-efficient algorithm with polylogarithmic span when the inputs are positive integers with word size $n^{\text{polylog}(n)}$. We present our theoretical results in Thm. 3.1, 3.2, 4.1, 4.2, 5.1 and 5.2. We believe this is the first paper that achieves near work-efficiency in parallel for a class of DP algorithms with DM.

Although the main focus of this paper is to achieve low work in theory, an additional goal is to make the algorithms simple and practical. We implement two algorithms as proofs-of-concept (code

available at [34]). On 10^9 input size, both of them outperform sequential solutions when the depth of the DP DAG is within 10^5 , and achieve 20–30 \times speedup with smaller depth of the DP DAG.

Due to page limit, we provide the full version of the paper in [33], which contains more algorithmic details, proofs, and some additional algorithms that fit into our framework.

Related Work. Dynamic programming (DP) is one of the most studied topics in algorithm design. The seminal survey [42] by Galil and Park reviewed two types of optimization techniques sequentially, including decision monotonicity (e.g., [35, 39, 41, 42, 60, 60, 61, 87]) and sparsity (e.g., [37, 38, 49, 54]). This paper mainly focuses on parallelizing the sequential algorithms in this scope, and we will review the literature of each problem in the corresponding section.

DP is also widely studied in parallel. There exists rich literature on optimizing various goals in different models, such as span (time) in PRAM (e.g. [6, 9, 11, 43, 52, 67, 68, 70, 71, 74, 78]), I/O cost in the external-memory/ideal-cache model (e.g., [25–28, 57, 81]), rounds in the MPC model [12, 20, 48, 55], or on the BSP model [5, 65, 66]. However, these papers either only considered the DP algorithms without the optimizations, or incur polynomial work overhead, except for [21, 23] for one specific problem. Instead, our paper tries to parallelize the efficient and practical sequential algorithms while maintaining low work. Some other works try to parallelize certain types of DP algorithms or applications using DP (e.g., [3, 10, 18, 32, 58, 73, 86]). Alternately, our work aims to provide a general approach to parallelize almost all DP algorithms.

2 MODEL AND FRAMEWORK

We use the **work-span model** in the classic multithreaded model with **binary-forking** [8, 15, 19]. We assume a set of threads that share the memory. Each thread acts like a sequential RAM plus a fork instruction that forks two threads running in parallel. When both threads finish, the original thread continues. A parallel-for is simulated by forks in a logarithmic number of steps. A computation can be viewed as a DAG. The **work** W of a parallel algorithm is the total number of operations, and the **span (depth)** S is the longest path in the DAG. In practice, we can execute the computation with work W and span S using a randomized work-stealing scheduler [19, 46] in time $W/P + O(S)$ with P processors with high probability. A parallel algorithm is **work-efficient**, if its work is $O(W)$, where W is the work of the best known or the corresponding sequential algorithm, and **nearly work-efficient** if its work is $\tilde{O}(W)$. We use $\tilde{O}(\cdot)$ to hide $\text{polylog}(n)$ where n is the input size.

2.1 Basic Concepts in Dynamic Programming

A DP algorithm solves an optimization problem by breaking it down to subproblems, memoizing the answers to the subproblems, and using them to find the answer to the original problem. The subproblems are usually indexed by integers, referred to as **states**. With clear context, we directly use i to refer to “state i ”. The DP value of a state is determined either by a **boundary condition** (i.e., initial values), or from other states, specified by a **DP recurrence**. This paper studies recurrences in the following form:

$$D[i] = \min/\max f_{i,j}(D[j]) \quad (1)$$

where j is a decision at i . Function $f_{i,j}(\cdot)$ indicates how the DP value of state j can be used to update state i . The transitions (i.e.,

dependencies) among states form a DAG $G = (V, E)$ as introduced in Sec. 1. We use $j \rightarrow i$ to denote an edge from j to i in the DAG, and use $best[i]$ to denote the best decision of state i .

During a DP algorithm, we may maintain the DP value of a state and update it by the recurrence. We call the process of updating $D[i]$ by $D[j]$ a **relaxation**, or say j **relaxes** i . A relaxation is successful if the DP value is updated to a **better** value, i.e., a lower (higher) value if the objective is minimum (maximum). We call the actual DP value of a state the **true** or **finalized** DP value to distinguish from the DP value being updated during the algorithm, which we call the **tentative** DP value. We say a state i is finalized if we can ensure that its true DP value has been computed, and tentative otherwise. Among all tentative states, we say a state is **ready**, if all its decisions are finalized, and **unready** otherwise. A naïve DP algorithm will process all transitions and states based on a topological ordering.

DP Optimizations. Instead of computing the recurrence straightforwardly, many algorithms can optimize the computation by skipping vertices and/or edges in the DAG to save work. We call such algorithms **optimized DP algorithms**. For example, given an input sequence $A[1..n]$, the optimized LIS algorithm [63] maintains a data structure to precisely find the best decision of each state in $O(\log n)$ cost, and only processes $O(n)$ transitions instead of $O(n^2)$ as suggested by the recurrence. Similarly, given two input sequences $A[1..n]$ and $B[1..m]$, the optimized LCS algorithm only needs to process all states $D[i, j]$ where $A[i] = B[j]$, instead of $O(nm)$ states in the recurrence. Another typical optimization is decision monotonicity where the best decisions of previous states can narrow down the range for best decisions for later states, which skips transitions and saves work. Typical examples include concave/convex generalized LWS (see Sec. 4), OAT (see Sec. 5.1), GAP (see Sec. 5.2), etc. We will discuss all these algorithms in this paper.

2.2 Parallelizing Sequential DP Algorithms

We now discuss our goal to parallelize a sequential DP algorithm. Our primary goal is to achieve (asymptotically) the same computation as an optimized sequential algorithm. More formally, for a sequential algorithm Γ computing a recurrence R with certain optimizations, we define the Γ -**optimized DAG**, denoted as $G_\Gamma = (V_\Gamma, E_\Gamma)$, as follows. G_Γ is the same as the DP DAG on R with some edges highlighted: for all edges that are processed by Γ , we highlight them in G_Γ , and call them the **effective edges**. We call the other edges **normal edges**. These effective edges can be used to fully complete the computation. For a sequential DP algorithm Γ , we hope its faithful and best possible parallelization Λ to:

- process the same effective edges in G_Γ and achieve the same work as Γ , and
- have span proportional to the **effective depth** of G_Γ , defined as the largest number of effective edges in any path in G_Γ .

Namely, if our goal is to parallelize Γ and perform the same computation, the parallel algorithm Λ has to process all edges $j \rightarrow i$ processed by Γ . This means i can be finalized only after j is finalized, so the span is related to the effective depth as defined above. We use \hat{E}_Γ to denote the set of effective edges, and $\hat{d}(G_\Gamma)$ as the effective depth of G_Γ . More formally, we define the following concepts.

We say a parallel algorithm Λ is an **optimal parallelization** of a sequential DP algorithm Γ , if 1) the set of edges processed in

Λ and Γ satisfies $\hat{E}_\Gamma \subseteq \hat{E}_\Lambda$ and $|\hat{E}_\Lambda| = O(|\hat{E}_\Gamma|)$, 2) the work of the two algorithms are asymptotically the same, and 3) the span of Λ is $\tilde{O}(\hat{d}(G_\Gamma))$. Namely, Λ can process a superset of edges of that in Γ , but not asymptotically more, and its span is proportional to the effective depth of the Γ -optimized DAG.

In addition, we define the **perfect parallelization**. In an omniscient version of Γ , we only need to process the best decisions based on their dependencies. We define the Γ -**perfect DAG**, denoted as G_Γ^* , as subgraph of G_Γ that only contains the best decision edges (both effective and normal edges). We say an optimal parallelization Λ of Γ is also a **perfect parallelization** if the span of Λ is $\tilde{O}(\hat{d}(G_\Gamma^*))$.

While the definitions seem abstract, we will later show that they are intuitive for concrete problems. For example, in longest common subsequence (LCS), both $\hat{d}(G_\Gamma)$ and $\hat{d}(G_\Gamma^*)$ are the output LCS length k (but for other algorithms they can be different), and our goal is to achieve $\tilde{O}(k)$ span for a perfect parallelization.

Note that the “perfect parallelization” of a sequential algorithm does not directly suggest optimality in work or span bounds for the same *problem*. One can possibly achieve better bounds by redesigning the recurrence and/or sequential algorithm with fewer edges or a shallower depth. Instead of finding or redesigning a different DAG to obtain new optimizations, our focus is to provide parallelization of existing sequential algorithms with optimizations.

In the following, we first present our new algorithmic framework: the Cordon Algorithm, which provides a correct, although not necessarily efficient parallelization for general DP algorithms. On top of it, for each specific problem, we will show how to achieve low work, which, as discussed in Sec. 4 and 5, can be highly non-trivial.

2.3 Our Framework: the Cordon Algorithm

Our idea is based on the *phase-parallel* framework [79] (see below) adapted to DP algorithms. The phase-parallel framework by Shen et al. aims to identify (as many as possible) operations that do not depend on each other, and process them in parallel. Directly applying this framework to DP algorithms will give the following algorithm outline:

Phase-Parallel Framework for DP algorithms

While there exist any tentative states:

- Find the set of ready states as \mathcal{F}
- Process all states in \mathcal{F} in parallel and mark all of them as finalized

We call each iteration of the while-loop a **round**. We call the set of states being processed in round i the **frontier** of round i , denoted as \mathcal{F}_i . While the phase-parallel framework gives a high-level approach in achieving parallelism, it does not indicate how to do so (i.e., how to identify the ready states in each round).

We now introduce our Cordon Algorithm, which uses a novel approach to identify the frontier \mathcal{F}_i in each round, particularly for a DP computation G_Γ . The Cordon Algorithm identifies the unready tentative states and put **sentinels** on them; then it uses all sentinels to outline a **cordon** to mark the boundary of the frontier. We summarize the algorithm in the following steps. Note that every step can be processed in parallel.

Step 1 Mark all states as tentative and initialize them by the boundary condition.

Step 2 If a tentative state j can successfully relax another tentative state i (i.e., update $D[i]$ to a better value), put a sentinel on state i . Such a sentinel means that all the descendants (inclusive) of state i are unready. We say this state and all its descendants are **blocked** by the sentinel in this case. Therefore, a state is **ready** if there is no sentinel on any of its ancestors (inclusive).

Step 3 For each ready state, use its DP value to relax the tentative DP values of its descendants. Usually, we need to do so *implicitly* to achieve efficient work. We discuss more details later.

Step 4 Mark all ready states as finalized. Clear all the sentinels.

Step 5 If there still exist tentative states, go to **Step 2** and repeat.

We will first prove that the algorithm is correct, i.e., it computes correct DP values for all states. Later we will show some motivating examples to help understand this algorithm in Sec. 3.

THEOREM 2.1. *The Cordon Algorithm is correct.*

Proof. This is equivalent to show that, when we find a ready state and later mark it finalized, its DP value must be finalized.

We will show this inductively. At the beginning, the ready states are those with zero in-degree in the DAG, and their true DP values are specified by the boundary cases. Clearly, their DP values cannot be relaxed by other states and they will be identified as ready in our algorithm. Their DP values are also computed by the boundary in **Step 1**. Therefore, our algorithm is correct in the first round.

Assume the algorithm is correct up to round r . We will show that round $r + 1$ also correctly finds the ready states and their DP values. Assume to the contrary that a state i identified in **Step 2** does not have its true DP value yet. This means that the best decision of i has not relaxed i to the finalized value. Let $j = \text{best}[i]$. Note that j cannot be finalized: if so, before j is marked as finalized in **Step 4**, in **Step 3** it must have relaxed i .

Therefore, j is tentative. In this case, let $j_0 = i$, and state j_x be the best decision of state j_{x-1} for $x \geq 1$, i.e., we chase the chain of best decisions and get a list of states $i = j_0, j_1 = \text{best}[j_0]$, etc. Let us find the first x such that $D[j_x]$ is not the true DP value but $D[j_{x+1}]$ is the true DP value. We then prove that state j_{x+1} must be a tentative state. If we consider j_y as the first finalized state on this chain, then j_{y-1} is a tentative state, and also must already have its true DP value (because j_y has relaxed it in **Step 3**). Note that since state j_{y-1} is a tentative state, all states between j_0 and j_{y-1} must be tentative. Since j_{y-1} has its true DP value, and $i = j_0$ does not have its true DP value, the first switch point j_x must be between j_0 and j_{y-1} , and must also be a tentative state.

Therefore, state j_{x+1} is a tentative state that can relax state j_x , so it will put a sentinel on state j_x . As a descendant of j_x , state i must be blocked by j_x , and cannot be identified as a ready state. This leads to a contradiction. Therefore, if a state i is identified as ready in our algorithm, its DP value must have been finalized. \square

The Cordon Algorithm tells which states/vertices should be in the frontier in each round, but the algorithm does not show how to do so *efficiently*. Especially in **Step 3**, it is almost infeasible to use the finalized DP values to explicitly update all other states. In this case, we have to develop new parallel data structures to facilitate this step. Next, we first use LIS and LCS as two examples to illustrate our framework. We then apply it to more involved cases that use decision monotonicity in Sec. 4 and 5.

3 MOTIVATING EXAMPLES ON LIS/LCS

To help the audience understand the more complicated algorithms using DM in the following sections, we first provide two simple examples on Cordon Algorithm, especially on how to compute the cordon efficiently.

Longest Increasing Subsequence (LIS). We first use LIS as an example. Given an input sequence A_i , LIS computes the maximum of $D[i]$ such that:

$$D[i] = \max\{1, \max_{j < i, A_j < A_i} D[j] + 1\} \quad (2)$$

We use n as the input size and k as the output LIS length. Directly computing this recurrence takes $O(n^2)$ work. Sequentially, we can process all states one by one, and compute $\max_{j < i, A_j < A_i} D[j]$ using a binary search structure, giving $O(n \log k)$ total cost [63]. The binary search precisely finds the best decision of each state, and only n transitions are processed. Many parallel LIS solutions have also been proposed [22, 45, 65, 66, 79]. We will show that solving LIS using our Cordon Algorithm framework will essentially give an existing algorithm [45] and is a perfect parallelization of the sequential $O(n \log k)$ algorithm.

Based on Cordon Algorithm, the boundary case is to set all tentative DP values as 1. Then we will attempt to use the current tentative DP values for relaxation. In this case, for a state i , as long as there is any other state $j < i$ with $A_j < A_i$, $D[i]$ can be relaxed to a better value 2. Therefore, all ready states are those input objects that are *prefix-min* elements in the sequence, i.e., A_i is the smallest value among all $A_{1..i}$. We can set these states as ready, update all other states and repeat. Note that since all unfinalized states have the same tentative DP value of 2, we *do not need to explicitly update* the values in $D[i]$, but can just maintain a global variable as the current tentative DP value. By the same idea, the ready states in the next round would be the prefix-min elements in the input after removing the finalized states. The same observation (repeatedly finding prefix-min elements) is exactly the core idea of the algorithm in [45]. In their algorithm, they further use a tournament tree to identify prefix-min elements efficiently and achieve efficient $O(n \log k)$ work, and $O(k \log n)$ span. Note that k is exactly the perfect depth of the DP DAG, which is the longest dependency between best decisions.

THEOREM 3.1. *Combining with a tournament tree, Cordon Algorithm leads to a perfect parallelization of sequential $O(n \log k)$ LIS algorithm in [45], where n is the input size, and k is the LIS length.*

Longest Common Subsequence (LCS). LIS has a close relationship with other important problems such as LCS. Here we will revisit our cordon-based LIS algorithm from the view of LCS, which also leads to a new parallel LCS algorithm. Given two sequences $A[1..n]$ and $B[1..m]$ ($m \leq n$), LCS aims to find a common subsequence C of A and B such that C has the longest length among all common subsequences. An LIS problem can be reduced to an LCS problem by first relabeling all input elements by $1..n$ based on their total order, and then finding the LCS between this new sequence and a sequence $B = \langle 1, 2, \dots, n \rangle$. See Fig. 2(a)–(c) for an illustration. LCS has been extensively studied both sequentially [7, 38, 49, 54, 63] and in parallel [6, 11, 24, 32, 74, 82, 88]. The standard DP solution defines each state $D[i, j]$ as the LCS for $A[1..i]$ and $B[1..j]$, and

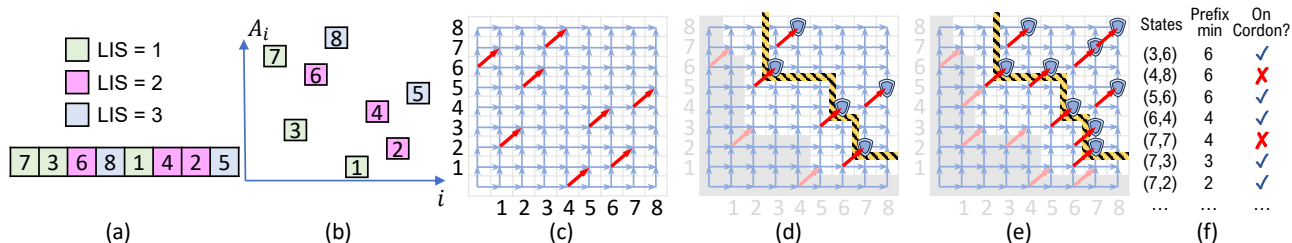


Figure 2: Illustrations for the LIS/LCS problem and the Cordon Algorithm. Subfigure (a): An input sequence for LIS with the DP value of each element. Subfigure (b): A geometric view of the input sequence on a 2D plane with each element represented as (i, A_i) . Subfigure (c): The corresponding LCS on this input—the answer is the longest path from $(0, 0)$ to $(8, 8)$ using the maximum number of red edges. Subfigure (d): The process to compute the second cordon. The ready states are marked in the shaded gray region. The cordon is decided by the three cells with LIS=2 in the original input. Subfigure (e): A general LCS case where every diagonal can be a red edge. Subfigure (f): An example execution of our LCS algorithm. Here we only show unready states for simplicity.

uses the following recurrence:

$$D[i, j] = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0 \\ D[i-1, j-1] + 1, & \text{if } A[i] = B[j] \\ \max\{D[i-1, j], D[i, j-1]\}, & \text{otherwise.} \end{cases} \quad (3)$$

These transitions correspond to horizontal, vertical and diagonal edges on a grid (see an example in Fig. 2(c)). A known sequential optimization (i.e., sparsification) [7, 38, 49, 54] to this recurrence is to observe that only the edges correspond to the diagonal edges with $A[i] = B[j]$ are useful. The computation is equivalent to finding the longest path from the bottom-left to top-right corresponds to these effective (red) edges, and all other edges and states can be skipped. This can lead to a sequential algorithm with $O(L \log n)$ cost, where L is the number of pairs (i, j) such that $A[i] = B[j]$. For LIS, there are exactly $L = n$ such effective edges.

We will show how Cordon Algorithm can be used to parallelize this optimization. Starting with the boundary where $D[i, j] = 0$, we observe that the DP value $D[i, j]$ of any state with $A[i] = B[j]$ can be updated to a *better* value. Therefore, we will put a sentinel at each of such states to indicate that they should be updated. All such sentinels will block the top-right part of the grid. In this way, the blocked region is clearly marked by a *staircase* region, as shown in Fig. 2(d). Therefore, the entire region within the first cordon has the DP value 0. By repeatedly doing this, we will effectively find that the region between the cordons of round $i+1$ and i are those states with DP value (LCS length) i . The algorithm finishes in k rounds where k is the LCS length.

The problem boils down to efficiently identifying the cordon (i.e., the staircase) in each round. Note that in LIS there is at most one effective edge in each column (see Fig. 2(d)), while in LCS there can be multiple effective edges in each column (see Fig. 2(e)). We will show an interesting modification to the original LIS algorithm that can handle this more complicated setting. Here we sort all edges by column index as the primary key (from the smallest to largest) and row index as the secondary key (but from largest to smallest). An example is illustrated in Fig. 2(f). Then, we will still use a tournament tree to maintain this list, and apply prefix-min on the row indexes. It is easy to see that a state/edge is on the cordon if its row index is smaller than or equal to the prefix-min. A tournament tree can identify, mark, and remove these states in $O(l \log(L/l))$ work and $O(\log n)$ span [45], where l is the number of diagonal edges on the cordon. We thus have the following theorem.

THEOREM 3.2. *Combining with a tournament tree, Cordon Algorithm leads to a perfect parallelization ($O(L \log n)$ work and $O(k \log n)$ span) of sequential LCS algorithm in [7], where n and $m < n$ are the input sequence sizes, L is the number of pairs (i, j) such that $A[i] = B[j]$, and k is the LCS length.*

Since $L = O(n^2)$, the $O(\log L)$ terms in the cost of the tournament trees is stated as $O(\log n)$ in the theorem.

Interestingly, to the best of our knowledge, this is the first parallel LCS algorithm with $o(mn)$ work and $o(\min(n, m))$ span for sparse LCS problem (i.e., $L = o(mn)$ and $k = o(\min(n, m))$). Meanwhile, this algorithm is quite simple—we provide our implementation in [34] and experimentally study it in Sec. 6. Another interesting finding is that our algorithm implies how to map LCS to LIS (we only know the other direction). Given two input strings A and B , if we sort all (i, j) pairs for $A[i] = B[j]$ by increasing i (primary key) and decreasing j (secondary key), then LCS is equivalent to the LIS on the secondary keys (the j (s)) of this sorted list.

We will show more sophisticated parallelization of DP algorithms in the next sections. Our LCS algorithm will be a subroutine in the more involved parallel GAP algorithm introduced in Sec. 5.2.

4 PARALLEL GENERALIZED LWS

We now discuss the convex/concave *generalized least weight subsequence (GLWS)* problem, which is one of the most classic cases of decision monotonicity (DM). Given a cost function $w(j, i)$ for integers $0 \leq j < i \leq n$ and $D[0]$, the GLWS problem computes:

$$D[i] = \min_{0 \leq j < i} \{E[j] + w(j, i)\} \quad (4)$$

for $1 \leq i \leq n$, where $E[j] = f(D[j], j)$ can be computed in constant time. The original least weight subsequence (LWS) problem [50] is a special case when $E[j] = D[j]$. Here we use the general case $E[j] = f(D[j], j)$ that has the same sequential work bound [35, 36, 39, 41, 42, 60], because the generalized version is needed in many applications (see examples in Sec. 5). The GLWS problem is also referred to as 1D/1D DP by Galil and Park [42]. The GLWS problem is highly relevant to other important problems (e.g., line breaking [64], optimal alphabetic trees [70], and a number of computational geometry problems [1]). The essence of GLWS is to cluster a list of 1D objects based on spatial proximity and minimize the total weighted sum of all clusters. As an intuitive example, consider selecting a subset of villages on a road (with their coordinates known) to build post offices to minimize the total cost,

where $w(j, i)$ is the cost of using one post office to serve villages $j + 1$ to i . This gives a GLWS problem with $D[i]$ as the lowest cost to serve the first i villages and $E[i] = D[i]$. The DP recurrence enumerates all possible decisions j such that the last post office serves the villages $j + 1$ to i , and takes the minimum cost among all possible decisions j . Many cost functions w used in practice (e.g., a fixed cost plus a linear or quadratic cost to the service range or sum of distances from villages to the post office) are *convex*, which implies decision monotonicity (DM) – for two states i and $j > i$, their best decisions i^* and j^* satisfy $j^* \geq i^*$. Symmetrically we can show that for *concave* cost functions w , either $j^* \geq i$ or $j^* \leq i^*$ holds, although concave cost functions are less common in the real-world applications of GLWS.

Given its high relevance in practice, convex GLWS has been studied in parallel. Apostolico et al. [6] showed an algorithm with $O(n^2 \log n)$ work and $O(\log^2 n)$ span. Later, Larmore and Przytycka [67] showed an improved algorithm with $O(n^{1.5} \log n)$ work and $O(\sqrt{n} \log n)$ span. Despite the interesting algorithmic insights in these algorithms, the polynomial overhead in work limits their potential to outperform the classic sequential solutions with $\tilde{O}(n)$ work [35, 36, 39, 41, 42, 60]. For the concave case, some work [21, 23] achieved near work-efficiency and polylog span on the original LWS, but the ideas cannot be applied to generalized LWS.

In this section, we show how to use the Cordon Algorithm to parallelize a well-known sequential GLWS algorithm with $O(n \log n)$ work, which works for both convex and concave DM. Although efficiently applying Cordon Algorithm here requires many sophisticated algorithmic techniques, our parallel algorithm (Alg. 1) remains practical and it indeed outperforms the sequential algorithm in a wide parameter range (see Sec. 6 for details). It is also the key building block for many other algorithms shown later in Sec. 5.

We start with preliminaries and the classic sequential algorithm, then discuss how to use Cordon Algorithm to parallelize it. We will use the convex case when describing the algorithm since it is used more often in practice, and discuss the concave case in Sec. 4.3.

4.1 Preliminaries

Convexity, Concavity and Decision Monotonicity. The convexity of the cost function w is defined by the Monge condition [76]. We say that w satisfies the **convex** Monge condition (also known as quadrangle inequality [89]) if for all $a < b < c < d$,

$$w(a, c) + w(b, d) \leq w(b, c) + w(a, d). \quad (5)$$

We say that w satisfies the **concave** Monge condition (also known as inverse quadrangle inequality) if for all $a < b < c < d$,

$$w(a, c) + w(b, d) \geq w(b, c) + w(a, d). \quad (6)$$

Consider two states i and $j > i$ with best decisions i^* and j^* . A convex weight function leads to DM such that $j^* \geq i^*$. A concave weight function leads to DM such that either $j^* \geq i$ or $j^* \leq i^*$.

A more general condition to enable DM is *convex/concave total monotonicity* [2]. Our algorithm works as long as the transition function is convex/concave totally monotone. For page limit, we provide the full background of related discussions in [33].

The Sequential Algorithm. The best (sequential) work bound for convex GLWS is $O(n)$ [42, 60, 72], and $O(n\alpha(n))$ for the concave case [61]. However, both of them are mainly of theoretical interest since they are complicated and have large constants in both work

and space usage. We parallelize a simpler and more practical algorithm with $O(n \log n)$ work [42]. This algorithm computes $D[1..n]$ in order. It implicitly maintains the best decision array $best[1..n]$. When the algorithm finishes computing $D[i]$, the algorithm updates $best[(i + 1)..n]$ using $D[i]$, then $best[j]$ ($j > i$) will be the best decision of state j among states 0 to i .

However, maintaining and updating this array of size n for n iterations require quadratic work. Observe that after computing $D[i]$, $best[(i + 1)..n]$ must be non-decreasing in the convex case, and must be non-increasing in the concave case [42]. Hence, the algorithm maintains a “compressed” version of $best[(i + 1)..n]$ by a list of triples $([l, r], j)$, which indicates that all states between l and r have best decision j , i.e., $\forall i' \in [l, r], best[i'] = j$. The list is maintained by a *monotonic queue*, which is a classic data structure based on double-ended queue, and is inherently sequential. In the i -th iteration, we can directly find the decision of state i from the queue. After obtaining $D[i]$, the monotonic queue can be updated in $O(\log n)$ amortized cost to consider i as a decision for all later states. In total, this algorithm costs $O(n \log n)$ work. Here we refer to the audience to the original paper for details of this algorithm. We will call this algorithm Γ_{lws} . Making use of DM, Γ_{lws} only processes transitions between each state i and its best decision. The DAG $G_{\Gamma_{lws}}$ for this algorithm includes normal edges $j \rightarrow i$ for all $j < i$, and exactly n effective edges $best[i] \rightarrow i$ for all states i .

Due to simplicity, this algorithm is usually the choice of implementation in practice. We will show a parallel version of this algorithm using Cordon Algorithm.

4.2 Parallel Convex GLWS

We first give the parallel algorithm of convex GLWS. We will use the “post-office” problem mentioned above as a running example to explain the concepts, but our algorithm works for general cases.

Following the idea of the phase-parallel algorithm, with the current finalized states, the goal is to find all ready states as the frontier, where their true DP values can be computed from the finalized ones. We will use our Cordon Algorithm to find the frontier in each round. Naïvely, the recurrence suggests that a state depends on all states before it. However, note that a state is essentially ready as long as its best decision has been finalized. For the convex case, we will use the fact as shown below.

FACT 4.1. *In convex GLWS, let $S = \{j : j > i \wedge best[j] \leq i\}$, which is the set of states with best decisions no later than state i ; then S is a consecutive range of states starting from $i + 1$.*

This is a known fact in the sequential setting (can be proved by induction). It suggests that the frontier of each round in the phase-parallel algorithm is a consecutive range of states. Based on this idea, we will maintain *now* as the last finalized state in each round. Then in the next round, ideally, the algorithm should find the cordon at *cordon*, where all states $[now + 1, cordon - 1]$ are ready and can compute their true DP value from (i.e., have their best decisions at) states no later than *now*. We show an example of the post-office problem to illustrate the phase-parallel framework in Fig. 3. Based on the discussions above, ideally, in round i , the ready states in the frontier are those where the best solution contains i post offices. This is because their best decision must have $i - 1$ post offices, and must be finalized in the previous round.

Algorithm 1: Parallel Convex GLWS

Input: problem size n , $D[0]$, cost function $w(\cdot, \cdot)$ **Output:** $D[1..n]$: the DP table**Maintains:** B : an sorted array storing triples of $([l, r], j)$, meaning that for all $l \leq i \leq r$, the current best decision $best[i] = j$

```
1  $now \leftarrow 0$ 
2 while  $now < n$  do
3    $cordon \leftarrow \text{FindCordon}(now)$ 
4    $\text{UpdateBest}(now, cordon)$ 
5    $now \leftarrow cordon - 1$ 
6 return  $D[1..n]$ 
7 Function  $\text{FindCordon}(now)$ 
8    $cordon \leftarrow n + 1$ 
9   for  $t \leftarrow 1$  to  $\log n$  do
10     $l \leftarrow now + 2^{t-1}$ 
11     $r \leftarrow \min(n, now + 2^t - 1)$ 
12    ParallelForEach  $j \in [l, r]$  do
13      Let  $best[j]$  be the current best decision of  $j$  recorded by  $B$ 
14       $D[j] \leftarrow E[best[j]] + w(best[j], j)$  // relax  $j$ 
15      Binary search in  $B$  and find
16       $s_j = \min\{i : E[j] + w(j, i) < E[best[i]] + w(best[i], i)\}$ ,
17      i.e.,  $i$  is the first state that can be successfully relaxed by  $j$ 
18       $cordon \leftarrow \min(cordon, \min_{l \leq j \leq r} s_j)$ 
19      if  $cordon \leq r + 1$  then break
20    return  $cordon$ 
21 Function  $\text{UpdateBest}(now, cordon)$ 
22   Tree  $T \leftarrow \text{FindIntervals}(now + 1, cordon - 1, cordon, n)$ 
23   Flatten  $T$  into array  $B$ 
24   Merge adjacent intervals with the same best decision in  $B$ 
25 Function  $\text{FindIntervals}(j_l, j_r, i_l, i_r)$  // use  $D[j_l..j_r]$  to update  $best[i_l..i_r]$ 
26   if  $i_l > i_r$  then return  $null$ 
27   if  $j_l = j_r$  then return a leaf node  $([i_l, i_r], j_l)$ 
28    $i_m \leftarrow (i_l + i_r) / 2$ 
29    $j_m \leftarrow \text{argmin}_{j_l \leq j \leq j_r} (E[j] + w(j, i_m))$ 
30    $x \leftarrow \text{node}([i_m, i_m], j_m)$ 
31   In Parallel:
32      $T_l \leftarrow \text{FindIntervals}(j_l, j_m, i_l, i_m - 1)$ 
33      $T_r \leftarrow \text{FindIntervals}(j_m, j_r, i_m + 1, i_r)$ 
34   return node  $x$  with left child as  $T_l$  and right child as  $T_r$ 
```

This high-level idea is presented in the main function of Alg. 1. Starting from $now = 0$, given the current finalized states $[0, now]$, we will find all ready states $[now + 1, cordon - 1]$ using the Cordon Algorithm, which essentially will find the cordon at $cordon$. We explain this part with more details in Sec. 4.2.1. Similar to the sequential algorithm, we also maintain a data structure B to store all triples $([l, r], j)$ in order, which indicates that all states between l and r have best decisions at j . This data structure is essential to guarantee the efficiency of finding the next frontier, and also has to be updated after each round with the new DP values (Line 4).

4.2.1 Finding the Cordon. To find the ready states in each round, we use the Cordon Algorithm. Namely, with all states up to now finalized, we can attempt to use the tentative states after now to update other tentative DP values. Once we find any j that can update i , we put a sentinel at i . Among all sentinels, the smallest (leftmost) one will give the final position of the cordon.

However, note that we cannot afford exhaustive checking for all pairs of states (j, i) . First of all, checking all possible $j > now$ may incur large overhead in work, since most of the later states are unready anyway. Ideally, the algorithm should check up to exactly the position of $cordon$, but this would be a chicken-and-egg problem. To handle this, our idea is to use *prefix-doubling*, a common idea in parallel algorithm design (e.g., [17, 31, 45, 47, 79]) to achieve work-efficiency and high parallelism. Here, prefix-doubling is used in function FindCordon in Alg. 1, which attempts to extend the cordon by a batch of 2^{t-1} states for increasing t in each substep t . If the entire batch is ready—i.e., no states in $[now + 1, now + 2^t]$ can be relaxed by each other, and all sentinels are outside the batch—we try a larger step and extend the cordon to $now + 2^{t+1}$. During the process, we will maintain $cordon$ as the leftmost sentinel so far. Once we find $cordon$ is inside the batch, it means that this batch is not fully ready. Therefore, the process stops and returns the current value of $cordon$ to the main algorithm.

Using prefix doubling, the parallel algorithm may check more states than the ready ones, but the number of “wasted” states is at most twice of the “useful” ones which will be finalized in this round. Hence, the total number of processed states is $O(n)$.

We then discuss the way to avoid checking all states $i > j$ when j puts sentinels. By DM, if j can successfully relax i , then j can also successfully relax all states $i..n$. Therefore, we only need to put a sentinel at the first such state i . Recall that we maintain all best decision triples in a data structure B in sorted order. By DM, we can simply binary search ($O(\log n)$ cost) in B to find s_j as the first tentative state that can be updated by j , and put a sentinel there.

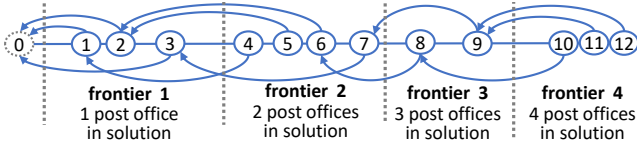
The FindCordon in Alg. 1 gives the full process as described above. Each iteration of the while-loop at Line 9 is a substep, which processes a batch of states in $[now + 2^{t-1}, now + 2^t]$ in substep t . Then for each state j in this batch (in parallel), we use B to find the first state that can be updated by j and put a sentinel at this position s_j . Finally, the leftmost sentinel so far forms the cordon. When the cordon is within the current batch, the algorithm returns. We also show an illustration of this process in Fig. 3.

LEMMA 4.1. *The function FindCordon has $O(h \log n)$ work and $O(\log^2 n)$ span, where $h = cordon - now - 1$ is the frontier size.*

Proof. As discussed above, the prefix doubling scheme may attempt to process up to h' states, where $h' \leq 2h$. For each such state, we may binary search in B to find s_j in $O(\log n)$ cost, and check the condition on Line 15 in $O(1)$ cost. Therefore, FindCordon has work $O(h \log n)$ and span $O(\log^2 n)$. \square

4.2.2 Generating New Best-Decision Array. The efficiency of the algorithm relies on maintaining an ordered data structure B for all best decision triples. We will store B as an array of all such triples in sorted order, such that the binary search in Line 15 can be performed efficiently. Therefore, after we get the newly finalized states $D[now + 1..cordon - 1]$, we need to update B accordingly to get the new best decision for all states in $[cordon, n]$.

We use a divide-and-conquer approach to do this. Function $\text{FindIntervals}(j_l, j_r, i_l, i_r)$ finds all best decision triples for states in range $[i_l, i_r]$, with best decisions in range $[j_l, j_r]$. Note that we only need the best decisions for all states after $cordon$. All these states must have their current best decisions within $[now + 1, cordon - 1]$ (if their best decisions are before now , they must have been ready in



Example of generating frontier 2

Subround 1 Process one state ④. $\text{cordon} = 13$.

Assume ④ cannot update any other tentative states.

Subround 2 Process next two states ⑤ and ⑥ in parallel.

Assume ⑤ cannot update any other tentative states.

The earliest state ⑥ can update is ⑧. So ⑥ puts a sentinel on ⑧. $\text{cordon} = 8$.

Subround 3 Process next four states ⑦-⑩ in parallel.

⑦ puts a sentinel on ⑨; ⑧-⑩ may or may not put sentinels on other states. $\text{cordon} = 8 < 10$. Return $\text{cordon} (= 8)$.

Figure 3: Example of applying the Cordon Algorithm to the post office problem with convex cost function. Circles (states) are villages. Arrows are best decisions between states. The final answer is four post offices serving villages 1–3, 4–7, 8–9, 10–12, respectively. The subrounds below illustrate the prefix-doubling scheme in FindCordon.

this round and been included in the frontier). Therefore, at the root level, we call $\text{FindIntervals}(\text{now} + 1, \text{cordon} - 1, \text{cordon}, n)$.

In FindIntervals , we first compute $j_m = \text{best}[i_m]$ where $i_m = (i_l + i_r)/2$, i.e., the best decision of the state in the middle. By (convex) DM, the best decisions of $i \in [i_l, i_m - 1]$ are in $[j_l, j_m]$, and the best decisions of $i \in [i_m + 1, i_r]$ are in $[j_m, j_r]$. We will deal with the two subproblems in parallel. To collect all $([l, r], j)$ triples in parallel, we build a tree-based structure bottom-up in the recursion. Finally, we flatten the tree to an array and merge the adjacent intervals if they have the same value of j .

LEMMA 4.2. *The function UpdateBest has $O(h \log n)$ work and $O(\log^2 n)$ span, where $h = \text{cordon} - \text{now} - 1$ is the frontier size.*

Proof. Flattening and removing duplicates can be performed by simple parallel primitives on trees and arrays in $O(h)$ work and $O(\log n)$ span. Below we will focus on the more complicated FindIntervals function. The span of FindIntervals comes from 1) $O(\log n)$ levels recursions and 2) $O(\log n)$ span to check all states in $[j_l, j_r]$ in parallel. For the work, each recursive call in FindIntervals deals with a range of states $[i_l, i_r]$ using best decision candidates in range $[j_l, j_r]$. The algorithm first finds $i_m \in [i_l, i_r]$ and its best decision $j_m \in [j_l, j_r]$. This can be done by comparing all possible decisions in $[j_l, j_r]$, which is $O(j_r - j_l)$ work. split the ranges into two subproblems and recurse. Let $N = |i_r - i_l + 1|$ and $M = |j_r - j_l + 1|$ denoting the sizes of the two ranges. The work of FindIntervals indicates the following recurrence:

$$W(N, M) = W(N/2, M_1) + W(N/2, M_2) + O(M)$$

where $M_1 + M_2 = O(M)$. This solves to $O(M \log n)$. On the root level, $M = \text{cordon} - \text{now} - 1 = h$. This proves that the work is $O(h \log n)$ for frontier size h . \square

4.3 Parallel Concave GLWS

To extend the algorithm to the concave case, we need a few modifications. In FindCordon , by the concavity, if j can update i , then j must be able to update $j + 1$. Therefore, in Line 15 in Alg. 1, we check whether j can update $j + 1$. If so, we put a sentinel at $j + 1$. The other modifications are in FindIntervals . First, due to concavity, when we find j_m as the best decision of i_m in Line 27, we need to

swap the last two parameters in the first and second recursive calls, i.e., the best decision range for states $i_m + 1$ to i_r must be before j_m , and those in i_l to $i_m - 1$ must be after j_m .

A more involved modification in the concave GLWS is that after we get the array B from FindIntervals , we have to merge it with the old array B before this round – FindIntervals only considers the best decisions among $[\text{now} + 1, \text{cordon} - 1]$, but in the concave case, these states may also have better decisions using states before now . For page limitation, we elaborate on this part in full paper [33]. With careful design, this part can also be finished in $O(h \log n)$ work and $O(\log n)$ span, where h is the frontier size.

4.4 Theoretical Analysis

In this section, we show the theoretical analysis for our parallel GLWS algorithm. We first summarize our main results as follows.

THEOREM 4.1. *Given an input sequence of size n , and the sequential GLWS algorithm Γ_{LWS} introduced in Sec. 4.1, let $k = \hat{d}(G_{\Gamma_{\text{LWS}}}^*)$ be the effective depth of the Γ_{LWS} -perfect DAG. Then the Cordon Algorithm for the convex GLWS has $O(n \log n)$ work and $O(k \log^2 n)$ span. It is a perfect parallelization of Γ_{LWS} .*

More intuitively, k in Thm. 4.1 is also the number of best decisions to make in the final solution: for instance, for the post-office problem, it is the number of post offices in the optimal solution.

THEOREM 4.2. *Given an input sequence of size n , and the sequential GLWS algorithm Γ_{LWS} introduced in Sec. 4.1, let $k = \hat{d}(G_{\Gamma_{\text{LWS}}})$ be the effective depth of the Γ_{LWS} -optimized DAG. Then the Cordon Algorithm for the concave GLWS has $O(n \log n)$ work and $O(k \log^2 n)$ span. It is an optimal parallelization of Γ_{LWS} .*

We first prove that both algorithms are nearly work-efficient and have $O(n \log n)$ work.

LEMMA 4.3. *The Cordon Algorithm for GLWS has $O(n \log n)$ work for both convex and concave case.*

Proof. Combining Lemma 4.1 and 4.2 (and the discussion in Sec. 4.3), the work for each round is $O(h \log n)$, where $h = \text{cordon} - \text{now} - 1$ is the frontier size. Since the frontier sizes h across all rounds add up to n , the entire algorithm has $O(n \log n)$ work. \square

We then show that the number of rounds in both convex and concave cases is the effective depth of $G_{\Gamma_{\text{LWS}}}$. Recall that the DAG $G_{\Gamma_{\text{LWS}}}$ includes normal edges between all states j and $i < j$, and effective edges between a state j and its best decision. The effective depth $\hat{d}(G_{\Gamma_{\text{LWS}}})$ is the largest number of effective edges in any path.

LEMMA 4.4. *The Cordon Algorithm for GLWS finishes in k rounds, where k is $\hat{d}(G_{\Gamma_{\text{LWS}}})$.*

Proof. Define the effective depth $\hat{d}(s)$ of a state s as the largest number of effective edges of a path ending at s . We will inductively prove that a state s is in the frontier of round r iff. s has effective depth r . The base case (boundary cases) holds trivially.

Assume the conclusion is true for $r - 1$. We first prove the “if” direction, i.e., if a state s has effective depth r , it must be in the frontier of round r . This is equivalent to show that there is no sentinel on all states from now to s . Assume to the contrary that there is a state $y \in (\text{now}, s]$ with a sentinel, which is put by state $x \in (\text{now}, y]$. This means that x is a better decision for y than all

states before *now*, indicating that y 's best decision $y^* \geq \text{now}$. Based on the induction hypothesis, the effective depth of y^* must be larger than $r - 1$. Therefore, $\hat{d}(y) = \hat{d}(y^*) + 1 > r - 1 + 1 = r$, which means that $\hat{d}(y)$ is at least $r + 1$. Based on the recurrence, there is a normal edge from y to s , so $\hat{d}(s) \geq r + 1$, leading to a contradiction.

We then prove the “only if” condition, i.e., if a state s is in the frontier of round r , it must have effective depth r . The induction hypothesis suggests that all states with effective depth smaller than r have been finalized in previous rounds, so we only need to show that $\hat{d}(s)$ cannot be larger than r . Assume to the contrary that $\hat{d}(s) \geq r + 1$. Let the path to s with effective depth $\hat{d}(s)$ be x_1, x_2, \dots, s . Since the total number of effective edges on this path is at least $r + 1$, there must exist an effective edge $x_i \rightarrow x_{i+1}$ on the path such that $\hat{d}(x_i) = r$ and $\hat{d}(x_{i+1}) = r + 1$. However, based on the induction hypothesis, x_i 's best decision must have been finalized. During Line 14, x must get its true DP value, and will find itself able to update x_{i+1} . Therefore, there will be a sentinel on $x_{i+1} \leq s$, and s cannot be identified in the frontier of round r . \square

We will then show that the number of rounds of the convex case is also the effective depth of the Γ_{LWS} -perfect DAG $G_{\Gamma_{\text{LWS}}}^*$. This is stronger than the Γ_{LWS} -optimized DAG as shown above. Recall that the perfect DAG $G_{\Gamma_{\text{LWS}}}^*$ contains all best decision edges in $G_{\Gamma_{\text{LWS}}}$.

LEMMA 4.5. *The Cordon Algorithm for convex GLWS runs in k^* rounds, where k^* is $\hat{d}(G_{\Gamma_{\text{LWS}}}^*)$.*

Proof. Define the perfect depth $d^*(s)$ of a state s as the largest number of effective edges of any path ending at s in $G_{\Gamma_{\text{LWS}}}^*$. Similarly, we will show by induction that in round r , all states with perfect depth r will be processed. The base case holds trivially. Assume the conclusion holds for $r - 1$. In round r , we will show that a state s with perfect depth r must be put in the frontier. Let s^* be the best decision of s , then $d^*(s^*) = r - 1$ and therefore $s^* < \text{now}$. According to DM, any state x between *now* and s must have its best decisions $x^* \leq s^* < \text{now}$, indicating that $d^*(x^*) \leq r - 1$. Therefore, x must find its true best decision in B , and cannot be updated by any other tentative states in Line 15. This means that there will be no sentinel between *now* and s , so s must be identified ready in round r . Therefore, a state with perfect depth r must be finalized in round r , leading to the stated theorem. \square

Combining Lemma 4.1, 4.2, 4.4 and 4.5 proves the span bounds in Thm. 4.1 and 4.2.

5 OTHER PARALLEL DP ALGORITHMS

We now show that our algorithmic framework can be used to parallelize a wide variety of classic sequential DP algorithms. Due to the space limit, we will focus on two of them and leave others in the full version of this paper [33], with a short summary in Sec. 5.3.

In particular, for the optimal alphabetic tree (OAT) problem (Sec. 5.1), we partially answered a long-standing open problem by Larmore et al. [70] for reasonable input instances (for instance, positive integer weights in range $n^{\text{polylog}(n)}$). For the GAP problem (Sec. 5.2), we showed the first nearly work-efficient algorithm with non-trivial parallelism. More interestingly, this algorithm combines all techniques in the algorithms for convex GLWS and sparse LCS.

5.1 Parallel Optimal Alphabetic Trees (OAT)

The optimal alphabetic tree (OAT) problem is a classic problem and has been widely studied both sequentially [30, 44, 51, 56, 59, 69, 77, 85] and in parallel [67, 68, 70, 78]. Given a sequence of non-negative weights $a_{1..n}$, the OAT is a binary search tree with n leaves and has the minimum cost, where the cost of a tree T is defined as:

$$\text{cost}(T) = \sum_{i=1}^n a_i d_i \quad (7)$$

Here d_i is the depth of the i -th leaf (from the left) of T (the root has depth 0). One can view the weight a_i as the frequency of accessing leaf i , and the depth of a leaf is the cost of accessing it. Then the cost of T is the total expected cost of accessing all leaves in T . The OAT problem is closely related to other important problems such as the optimal binary search tree (OBST) [62] and Huffman tree [53].

Sequentially, Hu and Tucker [51] showed an OAT algorithm with $O(n \log n)$ work. Later, Garsia and Wachs [44] simplified this algorithm. In parallel, Larmore et al. [70] showed an algorithm based on Garsia-Wachs. We will apply our techniques to this algorithm to improve the span bounds. Due to page limit, we provide the details of [70] in the full version paper [33], and review the high-level idea here. The algorithm computes an l -tree [44], which has the same depth with and will be finally converted to the OAT in $O(n)$ work and polylogarithmic span. The key insight of [70] is to start with a sequence of n leaf nodes with the input weights, and find several disjoint intervals in the sequence to process in parallel. This partition is done by various operations on the Cartesian tree of the input sequence, which requires $O(n \log n)$ work and $O(\log^2 n)$ span. Larmore et al. showed that processing each interval can be reduced to a convex LWS. The solution of the LWS will connect items in this interval into a forest, which becomes a subgraph in the final l -tree. Finally, for each tree in the forest, we insert its root back to the sequence and repeat the process. This reinsertion step takes $O(n \log n)$ work and $O(\log n)$ span by basic parallel primitives such as sorting and range-minimum queries. The further rounds will connect the forest to the final l -tree. Larmore et al. also showed that the number of such intervals shrinks to half in each iteration, so the algorithm will finish in $O(\log n)$ rounds. Here all other steps in addition to solving convex LWS take $O(n \log^2 n)$ work and $O(\log^3 n)$ span. The remaining cost of the algorithm is to solve convex LWS in each round, multiplied by the number of rounds, which is $O(\log n)$.

Larmore et al. originally used the parallel convex LWS algorithm from [4, 6], which has $O(m^2 \log m)$ work and $O(\log^2 m)$ span when taking an input interval of length m . Later, Larmore and Przytycka [67] improved the parallel convex LWS algorithm to $O(m^{1.5} \log m)$ work and $O(\sqrt{m} \log m)$ span, yielding $O(n^{1.5} \log^2 n)$ work and $O(\sqrt{n} \log^2 n)$ span for the OAT algorithm—the work overhead is still polynomial. Larmore et al. [70] left the open problem on whether there exists an OAT algorithm with $\tilde{O}(n)$ work and polylog(n) span, which remains unsolved for three decades.

Note that the convex LWS problem is a special case of the convex GLWS problem discussed in Sec. 4.1 with $E[i] = D[i]$. Hence, Alg. 1 directly gives $O(m \log m)$ work and $O(k \log^2 m)$ span for convex LWS problem, and here k is the longest dependency path of best decisions. In Larmore's algorithm, the forest for each interval is constructed iteratively by the DP algorithm on LWS: if iteration i finds the best decision at iteration $j < i$, then iteration i creates one

more level on top of the forest at iteration j . This means that the k is equivalent to the depth of the forest, which is upper bounded by the final OAT height h . We present more details in [33]. Hence, we can parameterize our final bounds using h as:

THEOREM 5.1. *The optimal alphabetic tree (OAT) can be constructed in $O(n \log^2 n)$ work and $O(h \log^3 n)$ span, where n is the size of input weight sequence and h is the height of the OAT.*

This algorithm is nearly work-efficient with span parameterized on h . One useful observation is that the OAT height h is polylogarithmic with real-world input instance with positive integer weights and fixed word length. More formally, we can show that:

LEMMA 5.1. *If all input weights are positive integers in word size W , the OAT height is $O(\log W)$.*

The proof is not complicated and given in the full paper [33]. With this lemma, we can state the following corollary.

COROLLARY 5.1.1. *If the input key weights are positive integers with word size $W = n^{\text{polylog } n}$, the OAT can be constructed in $O(n \log^2 n)$ work and $\text{polylog}(n)$ span, where n is the input size.*

The bounds also hold for real number weights if the ratio between the largest and smallest weight is $n^{\text{polylog } n}$. We note that in realistic models we usually assume word-size $W = n^{O(1)}$, in which case Cor. 5.1.1 affirmatively answers the open problem in [70].

5.2 The GAP Edit Distance Problem

The GAP problem is a variant of the famous edit distance problem. The GAP problem aligns two input strings with sizes n and $m \leq n$, and allows editing a substring with certain cost function (formally defined below). This problem has been widely studied both sequentially [26, 36, 40] and in parallel [16, 25, 28, 43, 57, 80, 83]. As noted by Eppstein et al. [36], most real-world cost functions are either convex or concave, yielding $\tilde{O}(nm)$ work for the GAP problem sequentially. Unfortunately, to the best of our knowledge, these existing parallel algorithms for the GAP problem need $\Omega(n^2 m)$ work, and the polynomial overhead makes them less practical.

More specifically, GAP takes two strings $A[1..n]$ and $B[1..m]$, and computes the minimum cost to align A and B using the following operations: 1) deleting $A[l+1..r]$ with cost $w_1(l, r)$, and 2) deleting $B[l+1..r]$ with cost $w_2(l, r)$. Here we consider the following recurrence, which is usually referred to as the GAP recurrence:

$$P[i, j] = \min_{0 \leq i' < i} D[i', j] + w_1(i', i)$$

$$Q[i, j] = \min_{0 \leq j' < j} D[i, j'] + w_2(j', j)$$

$$D[i, j] = \min\{P[i, j], Q[i, j], D[i-1, j-1] \mid A[i] = B[j]\}.$$

Here $P[i, j]$ and $Q[i, j]$ indicate the edits on the two strings. Directly computing the recurrence uses $O(n^2 m)$ work. Since most real-world cost functions in machine learning, NLP, and bioinformatics [36] are either convex or concave, sequentially each row in P or column in Q is a convex or concave GLWS and can be computed in $\tilde{O}(n)$ or $\tilde{O}(m)$ work. Hence, computing the entire P and Q takes $\tilde{O}(nm)$ work, leading to the same cost for computing D and the entire problem. We denote this standard sequential algorithm as Γ_{gap} .

Parallelizing this approach is extremely challenging even with the parallel convex/concave GLWS in Sec. 4 as a subroutine, and

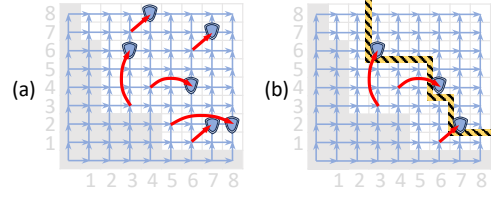


Figure 4: Example of a cordon in the GAP problem.

we are unaware of any existing work on this. The challenge here is that the rows in P interact with the columns in Q . For instance, computing a row in P requires one element from each column in Q , but computing those elements again requires previous rows in P .

Our key insight to parallelize this algorithm is to use the Cordon Algorithm to efficiently mark the ready region to be computed in each round. Note that as a generalization of the classic edit distance/LCS, the GAP recurrence is similar to Recurrence 3, but with “jumps” in computing P and Q . An illustration is given in Fig. 4. In addition to the diagonal edges as in LCS (see Fig. 2), for rows and columns, there also exist effective (red) edges (see Fig. 4(a)). Here for simplicity we only draw a subset of these edges, and every state $D[i, j]$ always have one vertical effective edge (to compute $Q[i, j]$), one horizontal effective edge (to compute $P[i, j]$), and may have a diagonal edge if $A[i] = B[j]$. All these edges imply the sentinels, which form the cordon and imply the regions for ready states, as shown in Fig. 4(b). The cordon is still a staircase as in LCS.

However, finding the cordon in GAP is sophisticated. We cannot directly use a tournament tree as in LIS, since the vertical and horizontal edges are computed on-the-fly and not known ahead of time. Meanwhile, in a 2D table where the cordon is a staircase, we cannot simply use prefix-doubling as in GLWS in Sec. 4. We propose a unique solution here to use prefix-doubling on a 2D table and computes the staircase cordon efficiently. This approach will consider each row separately, but for all rows, we run prefix doubling synchronously and try to see if the next ranges are available. First, we put a sentinel on state (x, y) with a diagonal edge if $(x-1, y-1)$ is not finalized. We will maintain the best-decision structure for each row and column, in the same way as the GLWS algorithm. For this region to be checked, we will use the same approach as in Alg. 1 to compute P and Q , take the minimum as D , and use D to check their readiness. If a state (x, y) obtains the best decision from another tentative state, we will put a sentinel on (x, y) , which will block the other states (x', y') with $x' \geq x$ and $y' \geq y$. The work to put the sentinels is proportional to the number of states we checked in the prefix-doubling, and the span is polylogarithmic.

Finally we discuss how to handle the sentinels placed as above. We store all sentinels based on the row index on increasing order. After this, applying a prefix-min on these sentinels gives part of the cordon (if they exist), and we will merge it with the previous cordon. Then, for all tentative states, we check whether they are on the correct side, and invalidate those across the cordon. Since we are using prefix doubling, the wasted work for the invalid states can be amortized. In the next prefix doubling step, we will also use the cordon to limit the search region. Once all states within the cordon are checked for readiness, we can move to the next round. Due to prefix doubling, we only need $O(\log n)$ steps in each round.

THEOREM 5.2. *The Cordon Algorithm for the GAP problem has $O(mn \log n)$ work and $O(k \log^2 n)$ span, where n and $m \leq n$ are the input size and k is the effective depth of the Γ_{gap} -optimized DAG for the sequential algorithm Γ_{gap} introduced in Sec. 5.2.*

Recall that the sequential GAP algorithm Γ_{gap} gets the DP value for each state $s = (i, j)$ by solving the GLWS problems in row i and column j , respectively, and the diagonal edge $(i - 1, j - 1) \rightarrow (i, j)$ if applicable. Therefore, the optimal DAG $G_{\Gamma_{\text{gap}}}$ contains three types of edges

- $(i, j) \rightarrow (i', j)$ for all $i' > i$,
- $(i, j) \rightarrow (i, j')$ for all $j' > j$, and
- $(i - 1, j - 1) \rightarrow (i, j)$ if $A[i] = B[j]$.

Among them, the effective edges include:

- $(i, j) \rightarrow (i', j)$ where i is the best decision for i' in the GLWS problem on row j ,
- $(i, j) \rightarrow (i, j')$ where j is the best decision for j' in the GLWS problem on column i , and
- $(i - 1, j - 1) \rightarrow (i, j)$ if $A[i] = B[j]$.

WLOG we assume $m \leq n$ in this section. We first prove the span bound. We will show that the Cordon Algorithm finishes in k rounds, where k is the effective depth of $G_{\Gamma_{\text{gap}}}$.

LEMMA 5.2. *Given two sequences of sizes n and $m \leq n$, the Cordon Algorithm on GAP edit distance finishes in $k = \hat{d}(G_{\Gamma_{\text{gap}}})$ rounds.*

Proof. The proof is similar to Lemma 4.4. We also define the effective depth of a state s as $\hat{d}s$. We will show by induction that s is processed in round r iff $\hat{d}s = r$. The base case holds trivially.

Assume the conclusion holds for all rounds up to $r - 1$. We will show it is also true for round r . We first prove the “if” direction, i.e., if a state $s = (i, j)$ (i -th row, j -th column) has effective depth r , it must be in the frontier of round r . This is equivalent to show that there is no sentinel that blocks s . For simple description, for two states $s = (i, j)$ and $s' = (i', j')$, we say $s < s'$ if $i \leq i'$ and $j \leq j'$. Clearly, if a state $s < s'$, a sentinel on s will block s' . Assume to the contrary that there is a state $y < s$ with a sentinel, which is put by another tentative state $x < y$. This means that the tentative state x is a better decision than all finalized states, indicating that the best decision of y , denoted as y^* , must also be tentative. Based on the induction hypothesis, the effective depth of y^* must be larger than $r - 1$. Therefore, $\hat{d}(y) \geq \hat{d}(y^*) + 1 > r - 1 + 1 = r$, which means that $\hat{d}(y)$ is at least $r + 1$. Let $y = (i', j')$. y and s can be connected by either one normal edge (when they are in the same row or column) or two normal edges $((i', j') \rightarrow (i, j')$ and $(i, j') \rightarrow (i, j))$. This means that the effective depth of s is at least the same as y , which is $r + 1$. This leads to a contradiction.

We then prove the “only if” condition, i.e., if a state s is in the frontier of round r , it must have effective depth r . The induction hypothesis suggests that all states with effective depth smaller than r have been finalized in previous rounds, so we only need to show that $\hat{d}(s)$ cannot be larger than r . Assume to the contrary that $\hat{d}(s) \geq r + 1$. Let the path to s with effective depth $\hat{d}(s)$ be x_1, x_2, \dots, s . Since the total number of effective edges on this path is at least $r + 1$, there must exist an effective edge $x_i \rightarrow x_{i+1}$ on the path such that $\hat{d}(x_i) = r$ and $\hat{d}(x_{i+1}) = r + 1$. However, based on

induction hypothesis, x_i 's best decision must have been finalized. In round r , x must get its true DP value, and will find itself able to update x_{i+1} . Therefore, there will be a sentinel on $x_{i+1} < s$, and s cannot be identified in the frontier of round r . \square

Combining Lemma 4.1 and 4.2, the span in each round is $O(\log^2 n)$. This proves the span bound in Thm. 5.2.

We then prove the work bound in Thm. 5.2.

LEMMA 5.3. *Given two sequences of sizes n and $m \leq n$, the Cordon Algorithm on GAP edit distance has work $O(mn \log n)$.*

Proof. As $G_{\Gamma_{\text{gap}}}$ is a grid graph, its depth is no more than $m + n$. By Lemma 5.2 the algorithm will finish in $k = O(n)$ rounds. In each round, we do prefix-doubling across all m rows and try to push the frontier on each row. In each prefix-doubling step we do a prefix-min that costs $O(m)$ work, so the cost of prefix-doubling is $O(m \log n)$ in each round, and $O(mn \log n)$ in total. Suppose h is the frontier size in one round. Due to prefix-doubling, the number of tentative states we visited is at most $2h$. Combining Lemma 4.1 and 4.2, in each row/column we can achieve work proportional to the number of tentative states. Thus the cost to put sentinels and maintain the best decision arrays is also $O(mn \log n)$. \square

5.3 Other Algorithms

We further apply the Cordon Algorithm to several other problems in the full version of this paper [33], including a GLWS problem on a tree structure, k -GLWS, and Optimal Binary Search Tree (OBST). For both k -GLWS and OBST, our algorithm leads to a correct algorithm with trivial parallelism ($\tilde{O}(k)$ and $\tilde{O}(n)$ span respectively). However, we show that they are still optimal parallelization of the sequential algorithms—this indicates that to further improve the parallelism, one should probably find or redesign another sequential algorithm to start with.

6 EXPERIMENTS

To demonstrate the practicability of our new algorithms, we designed experiments for LCS and convex GLWS. We implemented our parallel LCS algorithm and parallel GLWS algorithm in C++ using ParlayLib [14] to support fork-join parallelism and some parallel primitives (e.g., reduce). Our tests use a 96-core (192 hyper-threads) machine with four Intel Xeon Gold 6252 CPUs and 1.5 TB of main memory. We implemented two of our new algorithms (LCS and GLWS) as proofs-of-concept to show that our new techniques and algorithms remain practical due to work efficiency, which is also the main motivation of our paper. We release our code at [34], which gives more details of our experiments.

Parallel LCS. While as one of the classic algorithmic problem, LCS is widely studied in parallel, the existing parallel implementations we know of [6, 11, 24, 74, 82, 84, 88] do not take advantage of the sparsification. Most of them parallelizes the $\Theta(nm)$ algorithm for two strings with sizes n and m , and the experimental studies in these papers focus on input sizes up to 10^5 . They cannot process much larger instances in as the (sequential) sparse LCS algorithms (e.g., [7]) as discussed in Sec. 3. Hence, in our experiments, our baseline is our implementation of the sparse LCS in [7], which can process our input instances with large size (10^8) and small number of effective edges L (number of pairs (i, j) such that $A[i] = B[j]$).

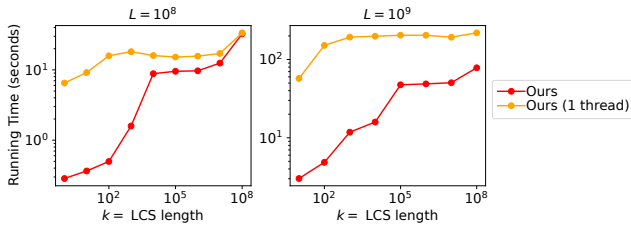


Figure 5: Running time of our parallel LCS algorithm (in Sec. 3).

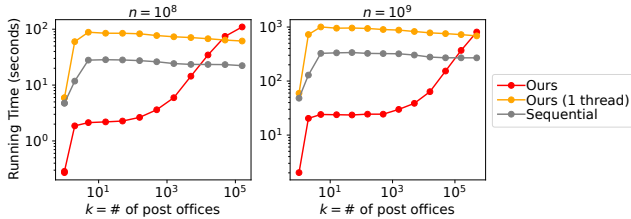


Figure 6: Running time of our parallel convex-GLWS algorithm (Alg. 1).

We test two random strings $A[1..n]$ and $B[1..n]$ with length $n = 10^8$, while controlling L and k (the LCS length). The pre-processing time to find all matching pairs is not counted into the running time. Fig. 5 shows the results when $L = 10^8$ and $L = 10^9$.

Overall, our algorithm has up to $30\times$ speed up than the sequential version. Since the span of the algorithm is proportional to the LCS length k , the running time increases when k increases. For $L = 10^8$, the parallel running time stays competitive to the sequential running time until the extreme case $k = 10^8$ (i.e., the two sequences are exactly the same). Since our algorithm has $O(L \log n)$ work and $O(k \log n)$ span, there is no parallelism when $L = k = 10^8$. For $L = 10^9$, where the total work is larger, our parallel algorithm is always faster than the sequential algorithm regardless of the value of k , and always achieves good parallelism.

Parallel GLWS. For GLWS, we use the setting of the post-office problem described in Sec. 4. We compare our parallel algorithm with the sequential solution in Sec. 4.1, as well as our own algorithm running on one core. We generate random data for $n = 10^8$ and 10^9 , and use different weight functions to control the output size k , which is the number of post offices in the solution. Fig. 6 shows the result on different n and k . The time for sequential algorithm does not change significantly, because it has $O(n \log n)$ work, which is independent of k . For our algorithm, the running time varies with k due to the $O(k \log^2 n)$ span. When k is small, our algorithm is $20\times$ faster than the sequential algorithm and achieves $30\text{--}40\times$ self-relative speedup. Our parallel algorithm is faster than the sequential algorithm for $k < 10^4$ when $n = 10^8$, and is faster than the sequential algorithm until $k \approx 10^5$ when $n = 10^9$.

7 CONCLUSION AND FUTURE WORK

We systematically studied general approaches to parallelize classic sequential dynamic programming algorithms, particularly those with non-trivial optimizations such as decision monotonicity and sparsification. We showed a novel framework, the Cordon Algorithm, and apply it to different DP recurrences. Theoretically, we gave the concept of optimal parallelism and perfect parallelism of a sequential algorithm, and showed that with a careful design, we

can achieve optimal parallelism for the classic sequential DP algorithms in a (nearly) work-efficient manner, and perfect parallelism for some instances. Practically, we show that our carefully-designed techniques do not include much overhead, and can outperform the original sequential version in a wide variety of cases.

We believe that the techniques in this paper opens a list of interesting questions. First, many of the new parallel algorithms are nearly work-efficient—we pick the most practical sequential algorithms for each problem, but they can be off the best work bound by up to an $O(\log n)$ factor. It is theoretically interesting to ask if we can match the best work bound in parallel. Second, among all these classic algorithms we looked at, one problem/algorithm that cannot be directly solved by the Cordon Algorithm is the RNA Secondary Structure [36]. Here, we may have 2^n different paths in a DP DAG, so applying Cordon Algorithm efficiently may need some complicated techniques. Finally, we show how to faithfully parallelize the sequential DP algorithms. We are aware of other approaches [6, 22, 66, 74, 82, 88] for LIS/LCS that can achieve stronger worst-case span bounds using divide-and-conquer. Hence, an interesting direction is to see if we can redesign other DP algorithms in a similar form to achieve better worst-case span.

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